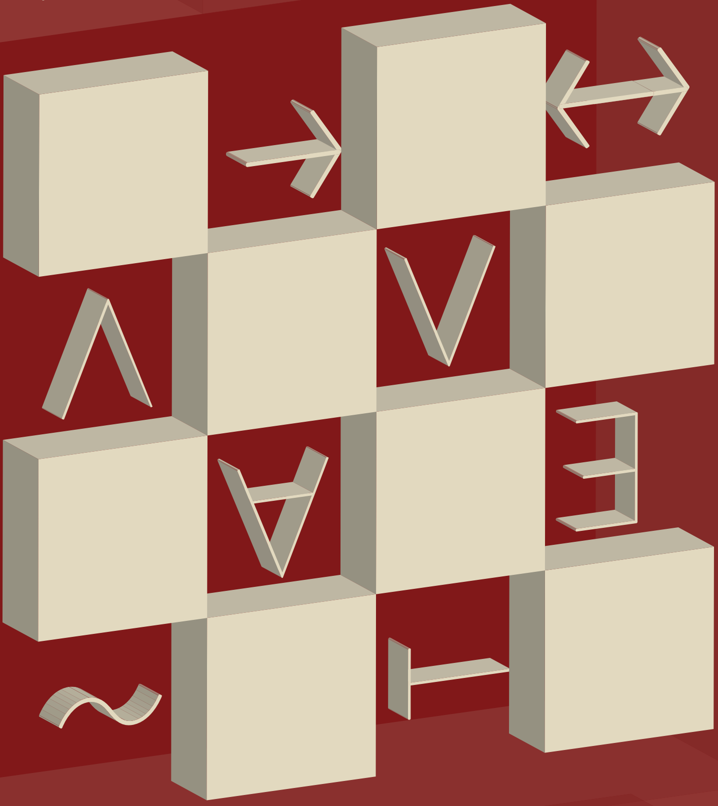
LOGIC, PROOF, and SETS

THIRD EDITION

Marvin L. Bittinger



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THIRD EDITION

Marvin L. Bittinger
Indiana University - Purdue University at Indianapolis

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DEDICATED TO THE MEMORY OF Nelson L. Zinsmeister

Nelson and I were roommates and fellow math majors at Manchester College. Two years my senior, Nelson was a most inspiring person who saw potential in me as a writer and math student at a time when I saw little in myself. Nelson went on to Purdue University attaining a Master's Degree in Mathematics, and accepted a position as a math teacher in upstate New York.

It was Nelson who stirred me to graduate studies in mathematics at The Ohio State University and then to Purdue University, where I received a PhD in Mathematics Education in 1968. I went on to teach at Indiana University – Purdue University at Indianapolis, became a math textbook author for Pearson Education, and retired as Professor Emeritus in Mathematics Education.

One night in the Fall, of 1963, Nelson, his pregnant wife Mary Martha, and her sister succumbed to a drunken driver who ran a stop sign south of Markle, IN. Etched in my heart is the pain of that news. I yearn to know how Nelson's life might have spawned numerous other students, teachers, and Christian leaders. I treasure his inspiration in my life.

PREFACE TO THE THIRD EDITION

LOGIC, PROOF, AND SETS by Marvin L. Bittinger

The purpose of this text is to provide a basic background in symbolic logic connected to mathematical proofs and attainable at an early level in the undergraduate curriculum. This text provides tools for the study of graduate mathematics.

There are various uses to which the book is addressed. The author has effectively used this material both as a supplement to the last semester of calculus, with an extra hour of credit given upon completion, as well as an introduction to a junior level course on the real number system, modern algebra, real analysis, linear algebra, or advanced calculus. It was found that such a study of logic, proof, and sets greatly speeded later study for the student knew how to form sentence negations, create proofs of conditional statements, proofs by contradiction, proofs by mathematical induction, proofs by cases, and so on.

The author wishes to express his appreciation to a number of people. Professor Angelo Margaris, formerly of The Ohio State University, taught a short unit in logic to a group of struggling graduate students and provided the bud of the idea to the author. His patience and that of others in the department allowed many of these students to go on to successful careers in mathematics.

This third edition could not have been accomplished without the dedication of two devotees of the book. Mike Rosenborg did extensive rewriting of the text and rendered the manuscript into new computer files. Phillip Lestmann of Bryan College did a final editing and prepared the files for online study. The author is in deep appreciation of their efforts.

This book was originally published in its First (1970) and Second (1982) editions by Addison-Wesley Publishing Co. The Third edition is published in 2017 by *2.718 ePub*, Inc. in Carmel, IN.

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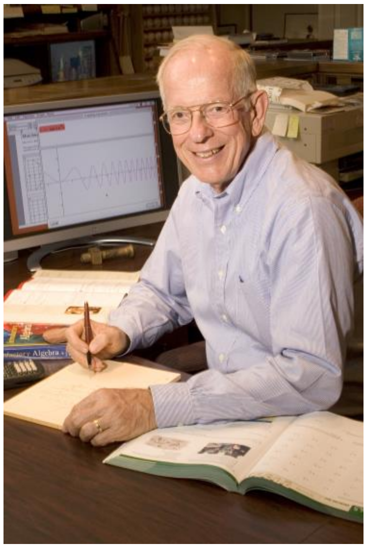
MLB

THE AUTHOR

Marvin L. Bittinger, is Professor Emeritus of Mathematics Education at Indiana University – Purdue University at Indianapolis.

Since earning his Ph.D. in mathematics education from Purdue University in 1968, Professor Bittinger has been teaching mathematics and writing textbooks at the university level for nearly 50 years. He has authored more than 250 mathematics publications which have sold more than 13 million copies. In addition, Marv has written a book entitled *Dusty Baker's Hitting Handbook,* co-authored by Jeff Mercer and longtime Major League Baseball player and manager Dusty Baker. He has also applied mathematics to the writing of a rather unique book, *The Faith Equation: Mathematical Evidence for Christianity* (for details see *www.faithequation.com*).



Professor Bittinger has also had the privilege of speaking at many mathematics conventions. His topics have included *Baseball and Mathematics, Mathematical Evidence for Christianity*, and numerous talks on the improvement of mathematics education at the college level.

Professor Bittinger's hobbies include baseball, softball, golf, and hiking in Utah. Marv truly loved his time at Purdue, and his fondness for the university has carried over to his family. He and his wife, Elaine; sons, Lowell and Chris; and granddaughters, Emma, Sarah, Maggie, and Claire; have traveled near and far to support the Boilermakers (aka, "Makers") in athletic competitions. In addition to watching Purdue softball games in Arizona and California, Marv also annually attends the Women's College World Series in Oklahoma City.

CHAPTER 1 LOGIC

INTRODUCTION

Just as an artist uses various tools and styles in his work, so does the mathematician. The purpose of this text is to study logic and to connect this to mathematical proofs; the tools of the mathematician.

A person can study impressionism, expressionism, and so on, can know all about water colors, oils, and canvas; but never be an artist. However, such knowledge does provide a firmer foundation for being an artist. Similarly, your study of this material does not guarantee you will be able to prove all you encounter, but should enhance your ability to do mathematical proofs.

1.1 SETS

This brief introduction to sets provides a basis for the study of logic in this chapter. In Chapter 3 we will study set theory more formally. Apart from notation this section could be omitted or lightly read by those previously exposed to the basic concepts of sets.

Symbolism. Braces are often used to name sets. For example, the set of integers 1, 2, 3, 4 could be named

$$\{1, 2, 3, 4\}.$$

This is the ***roster*** method for naming sets.

A method known as ***set-builder notation*** is often used to name sets. A property is specified which is held by all objects in a set. $P(x)$, read "P of x," will denote a sentence referring to the variable x. For example, "$x = 23$," "x is an even integer," and "$1 \le x \le 4$" are all sentences (or properties) referring to a variable x. The set of all objects x such that x satisfies $P(x)$ is designated

$$\{x \mid P(x)\}.$$

Thus, the set {1, 2, 3, 4} can be symbolized

$$\{x \mid 1 \le x \le 4, x \text{ is an integer}\},$$

which means

"The set of all x such that $1 \le x \le 4$ and x is an integer."

Membership. Henceforth, the words *object*, *element*, and *member* mean the same thing when referring to sets. For example, objects of sets are elements of sets and *vice versa*. The following have the same meaning:

$a \in A$,

a is in set A,

a is a member of set A,

a is an element of set A.

Similarly, $a \notin A$ means "a is **not** an element of set A."

EXAMPLES.

$1 \in \{1, 2, 3\}$

$5 \notin \{1, 2, 3\}$

$40 \in \{x \mid x \text{ is a multiple of } 10\}$

Subsets. *A* is a ***subset*** of *B* if every element of *A* is also an element of *B*. The following have the same meaning:

$A \subseteq B,$

Every element of *A* is an element of *B*,

If $a \in A,$ then $a \in B,$

A is included in *B*,

B contains *A*,

A is a subset of *B*.

EXAMPLES.

$\{1,2,3\} \subseteq \{1,2,3,4\}$
$\{1,3\} \subseteq \{1,3\}$

A set is always a subset of itself; that is, for any set *A*, $A \subseteq A.$ We prove this later. Also, $A \subseteq B$ means $B \supseteq A$ [read "*B* includes *A*"].

Equality for Sets. If *A* and *B* represent sets, then $A = B$ means that '*A*' and '*B*' represent the same set. The following have the same meaning:

$A = B,$

A and *B* name the same set,

A and *B* have precisely the same members,

$A \subseteq B$ and $B \subseteq A.$

EXAMPLES.

$\{1,2\} = \{x \mid (x-1)(x-2) = 0\}$
$\{\tfrac{1}{2}\} = \{x \mid 2x - 1 = 0\}$
$\{1,2,3\} = \{3,2,1\} = \{1,2,3,3\}$

Note that the order of listing elements is disregarded as well as repeated use of the same element.

The Empty Set. The set which contains no elements is known as the ***empty set*** and could be named { }, but we name it $\varnothing$. For any set *A*, $\varnothing \subseteq A.$ We prove this later.

Intersections. The ***intersection*** of two sets A and B is the set of elements common to both sets. The intersection is symbolized

$$A \cap B$$

or

$$\{x \mid x \in A \text{ and } x \in B\}.$$

EXAMPLES.

$\{1,3\} \cap \{1,2,3,4\} = \{1,3\}$
$\{1,3,5\} \cap \{1,2,3,\ldots\} = \{1,3,5\}$
$\{x \mid x > 1\} \cap \{x \mid x > 2\} = \{x \mid x > 2\}$
$\{2,4,6,8,\ldots\} \cap \{1,3,5,7,\ldots\} = \varnothing$

In the last example, there were no elements in common; so the intersection is the empty set.

Unions. The ***union*** of two sets A and B is the set of elements which are in A or B or both. The union is symbolized

$$A \cup B$$

or

$$\{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}.$$

EXAMPLES.

$\{1,2\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
$\{2,4,6,8,\ldots\} \cup \{1,3,5,7,\ldots\} = \{1,2,3,\ldots\}$
$\{x \mid x > 1\} \cup \{x \mid x > 2\} = \{x \mid x > 1\}$

Exercise Set 1.1

Use the roster method to name each set.

1. $\{x \mid x \text{ is an integer and } 1 \le x \le 8\}$
2. $\{y \mid y \text{ is an integer and } -5 < y < 0\}$
3. $\{x \mid x \text{ is an even integer}\}$
4. $\{x \mid x = 2k \text{ for some integer } k\}$

Use set builder notation to name each set.

5. $\{-1,0,1,2\}$

6. $\{10, 11, 12, 13, \ldots\}$
7. $\{10, 20, 30, 40, \ldots\}$
8. $\{3/2\}$

Place $\in$ or $\notin$ in each blank to make a true sentence.

9. 1_____ $\{2, 3\}$
10. 3_____ $\{5, 7, 3, 2\}$
11. 2_____ $\{x \mid x \text{ is an even integer}\}$
12. 15_____ $\{x \mid x \text{ is an even integer}\}$

Place $\subseteq$, $\supseteq$, or $=$ in each blank to make a true sentence.

13. $\{8, 9\}$ _____ $\{7, 11, 9, 8\}$
14. $\{5, 4, 3, 2, 1\}$ _____ $\{1, 2, 3\}$
15. $\{4, 5, 6\}$ _____ $\{6, 4, 5\}$
16. Pick out the pairs of sets which are equal.

$A = \{x \mid x^2 = 3,\ x \text{ even}\}$ $\quad$ $B = \{x \mid x^2 = 4\}$

$C = \{7, 2, 4\}$ $\quad$ $D = \{1, 2\}$

$E = \{8, 9, 7, 4\}$ $\quad$ $F = \{9, 9, 4, 7, 8\}$

$G = \{2, 1\}$ $\quad$ $H = \{-2, 2\}$

Determine whether true or false.

17. $\{x \mid x^2 = 3 \text{ and } x \text{ even}\} = \varnothing$
18. $\{1, 2\} = \varnothing$
19. $\{0\} = \varnothing$
20. $\{0\} \subseteq \varnothing$
21. $\varnothing \subseteq \{0\}$
22. $\varnothing \subseteq \{1, 2\}$

Find each of the following intersections.

23. $\{1/2, 1\} \cap \{-4, 8\}$
24. $\{3, 4, 5, 6, 7, \ldots\} \cap \{0, 1, 2, 3, 4\}$
25. $\{1, 2, 3\} \cap \varnothing$

26. $A \cap \varnothing,$ for any set A

27. $\{x \mid x < 0\} \cap \{x < -1\}$

Find each of the following unions.

28. $\{\tfrac{1}{2}, 1\} \cup \{-4, 8\}$

29. $\{3, 4, 5, 6, 7, \ldots\} \cup \{0, 1, 2, 3, 4\}$

30. $\{1, 2, 3\} \cup \varnothing$

31. $A \cup \varnothing,$ for any set A

32. $\{x \mid x < 0\} \cup \{x \mid x < -1\}$

1.2 UNIVERSAL SETS AND COMPLEMENTS

Universal Sets. Mathematicians always have a frame of reference called a ***universal set***. In plane geometry the universal set is the set of all points in the plane. In solid geometry the plane can no longer be used as the universal set. In calculus we consider the set of real numbers, the set of real functions, the set of differentiable functions, and the set of continuous functions as universal sets. Usually it is clear what the universal set is, though you may have to decide what it is from the title of the book or the chapter you are studying or from the context of the writing.

Complement. The ***complement*** of a set A is defined to be the set of all elements of the universal set which are not in A, and is symbolized

$$A'.$$

EXAMPLE. If a universal set

$$U = \{2,5,7,9,11,82\}$$

and

$$A = \{2,9,11,82\},$$

then

$$A' = \{5,7\}.$$

Note that $A \cup A'$ is always equal to the universal set, and $A \cap A'$ is always equal to $\varnothing$. We prove this later.

Subsets of the Real Numbers. We will use the following symbols to name specific subsets of the real numbers and refer to them as such throughout the book.

$\mathbb{N} = \{1,2,3,4,\ldots\} =$ The set of ***natural numbers***

$=$ The set of ***positive integers***

$\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0,1,2,3,4,\ldots\} =$ The set of ***nonnegative integers***

$=$ The set of ***whole numbers***

$$\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}= \text{ The set of } \textbf{\textit{integers}}$$

Notice that $\mathbb{N}\subseteq\mathbb{N}_0\subseteq\mathbb{Z}$.

$${}^{-}\mathbb{N}=\{\ldots,-3,-2,-1\}= \text{ The set of } \textbf{\textit{negative integers}}$$

P = {p | p > 1, p is a natural number, p is only divisible by 1 and p itself}

= The set of ***prime numbers***

Some elements of P are 2, 3, 5, 7, 11, 13, 17, and 19.

$$\mathbb{Q}=\left\{\tfrac{a}{b} \mid a\in\mathbb{Z} \text{ and } b\in\mathbb{Z} \text{ and } b\neq 0\right\}= \text{ The set of } \textbf{\textit{rational numbers}}$$

EXAMPLES. $\frac{2}{3}\in\mathbb{Q}$, $\quad -\frac{5}{4}\in\mathbb{Q}$

Notice $\mathbb{N}\subseteq\mathbb{N}_0\subseteq\mathbb{Z}\subseteq\mathbb{Q}$, since $\dfrac{a}{1}=a$ for any integer a.

J = {x | x cannot be expressed as a ratio of two integers}

= The set of ***irrational numbers***

EXAMPLES.

$\sqrt{2}\in J$

$\pi\in J$

$e\in J \quad (e=2.718\ldots)$

$2\notin J$

$3/4\notin J$

In fact, $\mathbb{Q}\cap J=\varnothing$.

$\mathbb{R}$ = $\mathbb{Q}\cup J$ = The set of ***real numbers***

Exercise Set 1.2

1. If a universal set $U=\{0,1,2,3,10,8\}$, $A=\{1,2,3,10\}$, $B=\{0,1,8,10\}$, and $C=\{0,1,2,3\}$, find the following.
   a) $A\cap(B\cup C)$
   b) A'
   c) $(A\cap B)'$
   d) $(A\cap B\cap C)'$

e) $(A \cap (B \cup C))'$

f) $A' \cup B'$

g) $(A \cap B) \cap C$

h) $A \cap (B \cap C)$

i) $(A \cup B) \cup C$

j) $A \cup (B \cup C)$

If $\mathbb{R}$ is the universal set, find:

2. $\mathbb{Q} \cap \mathbb{R}$
3. $\mathbb{Z} \cap J$
4. $\mathbb{N} \cap J$
5. $\mathbb{N}_0 \cap J$
6. $\mathbb{Z} \cap \mathbb{N}$
7. $\mathbb{Q}'$
8. $\mathbb{N}_0 \cup {}^{-}\mathbb{N}$
9. J'
10. Depict { x | x is an odd integer } in two other ways.
11. Suppose D = { x | x is an odd integer } and E = { x | x is an even integer }. Find:
    a) $D \cap E$
    b) $D \cup E$
12. Display the following sets by the roster method.
    a) $\mathbb{Z}_0 = \{x \mid x \in \mathbb{Z} \text{ and } x = 3k \text{ for some } k \in \mathbb{Z}\}$
    b) $\mathbb{Z}_1 = \{x \mid x \in \mathbb{Z} \text{ and } x = 3k + 1 \text{ for some } k \in \mathbb{Z}\}$
    c) $\mathbb{Z}_2 = \{x \mid x \in \mathbb{Z} \text{ and } x = 3k + 2 \text{ for some } k \in \mathbb{Z}\}$

    Then find:

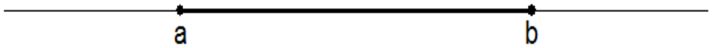
    d) $\mathbb{Z}_1 \cap \mathbb{Z}_0$
    e) $\mathbb{Z}_1 \cap \mathbb{Z}_2$
    f) $\mathbb{Z}_0 \cap \mathbb{Z}_2$
    g) $\mathbb{Z}_0 \cup \mathbb{Z}_1 \cup \mathbb{Z}_2$

Use the roster method to describe the sets in Exercises 13-18.

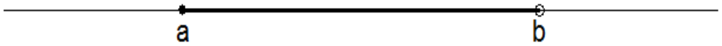
13. $\left\{x \mid x^2 + 2x + 1 = 0\right\} \cup \left\{x \mid x^2 + 4x + 4 = 0\right\}$
14. $\left\{x \mid x \in \mathbb{R} \text{ and } x^2 = -1\right\}$
15. $\{x \mid x \in J \text{ and } x \in \mathbb{Q}\}$

16. $\{x \mid x$ is an integer and x is not even$\}$
17. $\{x \mid x$ is an integer and x is not odd$\}$
18. $\{x \mid x$ is an even integer and x is prime$\}$
19. The intervals are important subsets of $\mathbb{R}$. They are defined and described on the real line as follows:

$[a,b]=\{x \mid a \le x \le b\}$

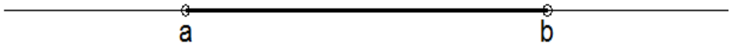


$[a,b)=\{x \mid a \le x < b\}$

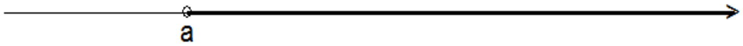


$(a,b]=\{x \mid a < x \le b\}$

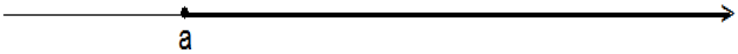
$(a,b)=\{x \mid a < x < b\}$



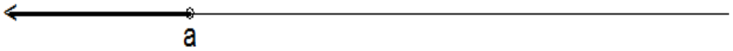
$(a,\infty)=\{x \mid a < x\}$



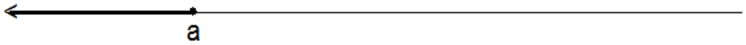
$[a,\infty)=\{x \mid a \le x\}$



$(-\infty,a)=[a,\infty)'$



$(-\infty,a]=(a,\infty)'$



Find:

a) $(-\infty,3)\cap[2,\infty)$
b) $(-\infty,3)\cup[3,\infty)$
c) $[-1,2)\cup[1,4)$
d) $[-1,2)\cap[1,4)$
e) $[3,3]$
f) $(3,3)$
g) $[-n,n]\cap[-(n+1),n+1]$
h) $[-n,n]\cup[-(n+1),n+1]$

20. If U is a set, then $\wp(U)$, the ***power set*** of U, is $\{A \mid A \subseteq U\}$. For example, $\wp(\{a,b\})=\{\varnothing,\{a\},\{b\},\{a,b\}\}$. Note that $\wp(U)$ is a set whose elements are sets. Find:

a) $\wp(\{1,2\})$
b) $\wp(\{0\})$

c) $\wp(\{1, 2, 3\})$

d) $\wp(\varnothing)$

e) Make a conjecture as an answer to the following question, based on the results of a)-d) above: The power set of a set with *n* elements has how many elements?

1.3 SENTENCES AND STATEMENTS

Logic and mathematical proof can be studied just as algebra, geometry, or calculus. To study logic is to study the language of mathematics.

Just as everyone uses sentences to convey ideas, mathematicians use sentences to convey their ideas; for example,

$$x+y=1,$$
$$2+3=5,$$
$$\int_0^3 \cos x\, dx = \sin 3,$$

and so on.

Statements. Declarative sentences which are true or false, but not both, are called *statements*. The following are statements.

Barry Bonds hit 73 home runs in one season. (True)

2 + 3 = 6 (False)

For every x, if $f(x)=\sin x$, then $f'(x)=\cos x$. (True)

The 26,000th digit of π is 4.

In the last example, we know it is a statement, though offhand we do not know whether it is true or whether it is false. The following are not statements.

Why are you studying mathematics?

He is a baseball player.

$x + 1 = 0$

$k - m = b$

Variables. The sentence,

He is a baseball player,

cannot be judged true or false because we do not know who ‘He’ is. If the word ‘He’ is replaced by ‘Donald Trump’ forming the sentence

Donald Trump is a baseball player,

The sentence becomes a (false) statement. Similarly, if ‘x’ in the sentence

$x + 1 = 0$

is replaced by ‘3’, forming the sentence

$\mathbf{3} + 1 = 0$,

The sentence then becomes a (false) statement.

The letter ‘x’ is a ***variable*** in the sentence $x + 1 = 0$. A letter (or other symbol) that can represent various elements of a universal set is called a ***variable***. Thus, ‘He’ is a variable in the sentence

He is a baseball player.

We can make a sentence a statement by replacing its variables by numbers or by attaching phrases such as “For every” or “There exists” to the sentence. For example,

$x < 3$

is not a statement, but each of the following is a statement:

$1 < 3$,

$5 < 3$,

For every real number x, $x < 3$,

There exists an x such that $x < 3$.

Solution Sets. Replacements for variables of a sentence are always chosen from some universal set.

EXAMPLE. Replace the variable in the sentence

$$x + 1 < 3$$

by each element of the universal set {0, 1, 2, 3} and decide the truth value of each resulting sentence.

$0 + 1 < 3$ (True)

$1 + 1 < 3$ (True)

$2 + 1 < 3$ (False)

$3 + 1 < 3$ (False)

Any replacement which makes a sentence true is called a ***solution***. The set of all solutions is called the ***solution set*** of the sentence. In the above example the solution set is {0, 1}.

You probably know other more direct ways of finding solution sets using algebra.

From now on we consider only sentences which are statements or statement forms which become statements when meaningful replacements are made for all their variables.

When doing proofs, we may consider a sentence like

$$\text{If } x = 11\text{, then } 3x = 33$$

to be a statement because we have assumed x represents an element of a universal set.

Exercise Set 1.3

Consider the following sentences for Exercises 1-3.

a) $x < 2$
b) $\lim\limits_{n\to\infty} n = 1$
c) $x + y = y + x$
d) There exists a natural number x such that $x < 2$.
e) For every real number x and every real number y, $x + y = y + x$.
f) $1 < 2$
g) $2 + 3 = 3 + 2$
h) This sentence is false.

1. Which of the above are statements?

2. Identify the variables in each sentence.

3. Which will become statements when the variables are replaced by numbers?

Find the solution set of each sentence with indicated universal set.

4. $x - 2 < 3$ {0, 1, 2, 3}

5. $|x| + 1 < 3$ {0, 1, 2, 3}

6. $(x-1)(x+2) = 0$ {5, 6, 7}

7. $(x-1)(x+2) = 0$ $\{-2, -1, 0, 1, 2\}$

8. $x^2 + 2x + 1 = 0$ $\mathbb{N}$

9. $x^2+2x+1=0$ $\qquad \mathbb{Z}$

10. $2x^2+3x+1=0$ $\qquad \mathbb{N}$

11. $2x^2+3x+1=0$ $\qquad {}^{-}\mathbb{N}$

12. $2x^2+3x+1=0$ $\qquad \mathbb{Z}$

13. $2x^2+3x+1=0$ $\qquad \mathbb{Q}$

14. $2x^2+3x+1=0$ $\qquad \mathbb{R}$

15. $x^2+1=0$ $\qquad \mathbb{R}$

16. $\frac{x^2-4}{x+2}=x-2$ $\qquad \mathbb{R}$

Decide the truth value of each of the following. Refer to a calculus book where appropriate. Assume that *x* represents a real number, and *f* represents a real function.

17. For every real number *x*, $x^2=0$.

18. If $x = 3$, then $x < 2$.

19. If $f(x)=x^2$, then $f'(x)=2x$.

20. If $x = 0$ or $x = 1$, then $x^2 = x$.

21. For every natural number *x*, $x^2 = x$.

22. There exists a natural number *x* such that $x^2 = x$.

23. $\sqrt{x^2}=|x|$.

24. If $|x|<3$, then $-3<x<3$.

25. Every rational number can be expressed as a ratio of two integers.

26. If $f(x)=|x|$, then *f* is continuous but not differentiable at $x = 0$.

27. The series $\sum_{n=1}^{\infty}(-1)^n n^{-1}$ is convergent but not absolutely convergent.

28. If a series is absolutely convergent, then it is convergent.

29. A series either converges or diverges.

30. If a function is continuous, then it is differentiable.

31. $\lim_{n \to 0} n^{-1} = 1$

32. For every real number x and every real number y, $x + y = 0$.

33. If a function is differentiable, then it is continuous.

34. There exists a real number x such that $x < 2$.

1.4 SENTENCE CONNECTIVES

Conjunction. If *P* and *Q* are sentences, then the sentence '*P* and *Q*' is called the ***conjunction*** of *P* and *Q*, symbolized

$$P \wedge Q.$$

For any statement there are just two possible truth values, true (T) or false (F). If *P* and *Q* are both true, then $P \wedge Q$ is true. If one or both of *P* and *Q* are false, then $P \wedge Q$ is false. The truth table below defines the truth values of $P \wedge Q$ for all possible truth value combinations of *P* and *Q*.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLES. $(2+2=4)\wedge(3+2=7)$ (False)

$(\pi \text{ is irrational})\wedge(\pi>0)$ (True)

Disjunction. If *P* and *Q* are sentences, then the sentence '*P* or *Q*' is called the ***disjunction*** of *P* and *Q*, symbolized

$$P \vee Q.$$

Unlike conjunctions there are at least two uses of *or* in English.

One use is exclusive, meaning "one or the other but not both." For example, the sentence

Are you awake or asleep?

cannot be answered *yes* because you cannot be both awake and asleep at the same time.

Another use is inclusive, meaning "and/or." For example, the sentence

Are you wearing a shirt or sweater?

could be answered *yes*. This would mean the answerer was wearing either a shirt, a sweater, or both.

The mathematician defines *or* to be inclusive; that is $P \vee Q$ is true when P is true, Q is true, or both are true. $P \vee Q$ is false only when P and Q are both false. The truth table for $P \vee Q$ is thus defined below.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLES. "$(2+2=4) \vee (3+2=5)$" is true because both are true.

"$(\pi \text{ is rational}) \vee (\pi \text{ is irrational})$" is true because "$\pi$ is irrational" is true.

Negation. A ***negation***, or denial, of a sentence may be formed in many ways. For example, the negation of

P: 2 is rational,

is represented by each of the following:

$\sim P$,

It is false that 2 is rational,

2 is not rational,

2 is irrational.

The truth table for negation is defined below.

P	$\sim P$
T	F
F	T

There are other symbols, besides ~, for negations. For example,

$a \neq b$ means $\sim (a = b)$,

$a \geq b$ means $\sim (a < b)$, and

$a \notin A$ means $\sim (a \in A)$.

Conditional. If P and Q are sentences, the sentence

If P, then Q

is symbolized

$$P \to Q.$$

The mathematician defines a truth table for $P \to Q$ just as he does for $\sim$, $\wedge$, and $\vee$; but the definition is not at all obvious. An example may help before we give the definition. Consider the sentence

If I get an A in mathematics, then I will take the next course.

Suppose a fellow student says this. When is he telling the truth and when is he lying? Examine the following four cases where

P means "I will get an A in mathematics"

and

Q means "I will take the next course."

1) P (true): He gets an A in mathematics

Q (true): He takes the next course

2) P (true): He gets an A in mathematics

Q (false): He does not take the next course

3) P (false): He does not get the A

Q (true): He takes the next course

4) P (false): He does not get the A

Q (false): He does not take the next course

In (1) it is reasonable to agree that the student was telling the truth; his claim is true. In (2) it is easy to agree that he lied, and his claim was false. In (3) you could not call him a liar since he takes the next course even though he did not get an A. In (4) you likewise could not call him a liar since he did not get the A and did not take the next course.

	P	Q	$P \to Q$
1)	T	T	T
2)	T	F	F
3)	F	T	T
4)	F	F	T

(The numbers refer to the preceding explanation.)

The truth table definition for $P \to Q$ conforms to the previous example. Should you be troubled about the definition, be comforted by the fact that mathematicians struggled with it for a long time. Any more explanation would repeat an example like the one above. Here is one place in mathematics where, if you are troubled, the easiest way out is to accept the definition and go on. (In English, the if-then sentence is used only when there is some logical or causal connection between the antecedent and the consequent, but in symbolic logic it is used without any such limitations. For example, the sentence "if the moon is made of green cheese, then I have two eyes" is true, although such a sentence is hardly appropriate or sensible in spoken or written English.)

The sentence $P \to Q$ is called a ***conditional*** with

P the ***antecedent***

and

Q the ***consequent***.

To summarize: A conditional is true when the antecedent is false or the consequent is true. A conditional is false only when the antecedent is true and the consequent is false.

In mathematics $P \to Q$ is encountered in many forms. You *should* be familiar with each. The following have the same meaning:

$P \to Q$,

If *P*, then *Q*,

P implies *Q*,

Q if *P*, *P* only if *Q*,

Q provided *P*,

Q whenever *P*, *Q* when *P*,

P is a sufficient condition for Q,

Q is a necessary condition for P.

(Memorize these)[*]

One meaning of

$$P \to Q$$

is

P is sufficient for Q.

This can be explained via its truth table. When P is true and $P \to Q$ is true, then Q must be true. In other words, P being true is enough (is sufficient) to yield Q being true when $P \to Q$ is true.

Another meaning of

$$P \to Q$$

is

Q is a necessary condition for P.

This is also explained via the truth table for $P \to Q$. If $P \to Q$ is true and Q is false, then P must be false; that is, if Q is false, so is P. Q being false *necessitates* P being false.

EXAMPLES. Translate to the form $P \to Q$.

a) A polygon has no diagonals only if it is a triangle.

Using the following translations:

P: A polygon has no diagonals,

Q: It is a triangle,

the sentence translates to a sentence of the type $P \to Q$, or

If a polygon has no diagonals, then it is a triangle.

b) The function *f* is continuous when it is differentiable.

Using the following translations:

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[*] The moral of this story is "Watch your P's and Q's!"

P: A function is differentiable,

Q: A function is continuous,

the sentence translates to the type $P \to Q$, or

If a function is differentiable, then it is continuous.

Experience at recognizing sentences with "If…then" structure, though not stated as such, will aid mathematical reading and proof.

Exercise Set 1.4

Find the truth value.

1. $\left(\lim_{n\to\infty}\frac{1}{n}=1\right)\wedge\left(e \text{ is rational}\right)$

2. $\left(\text{For every } x, \sqrt{x^2}=|x|\right)\wedge\left(4\neq 3\right)$

3. (π is rational) $\vee$ (π is real)

4. (π is an integer) $\vee$ (π is a natural number)

5. $(J\cap\mathbb{Q}=\mathbb{R})\vee\left(\int\sin x\,dx=\cos x+C\right)$

6. Let an infinite series $S=\sum_{n=1}^{\infty}u_n$; (S converges) $\vee$ (S diverges)

Write four different representations of the negations of each.

7. P: $2 = 3$

8. P: e is irrational

Find the truth value of each.

9. $2\neq 3$

10. ~ (e is irrational)

Give an expression for each of the following which does not involve a negation symbol.

11. ~ $(x < y)$

12. ~ $(x > y)$

13. $\sim (3 \le y)$

14. $\sim \left(z^2 \ge 1 + x\right)$

Find the truth value.

15. $2 < 1 \rightarrow 2 < 3$

16. $3 > 4 \rightarrow 6 < 5$

17. $2 = \sqrt{4} \rightarrow \sum_{n=1}^{\infty} n^{-1}$ converges.

18. $2 \ge 0 \rightarrow \sum_{n=1}^{\infty} n^{-2}$ converges.

Translate each sentence to the type "If *P*, then *Q*" and $P \rightarrow Q.$ Identify the antecedent and the consequent.

19. There is no factorization of *n* whenever *n* is prime.

20. $\sum_{n=1}^{\infty} u_n$ converges only if $\lim_{n \to \infty} u_n = 0.$

21. $|x| < 1$ implies that $\lim_{n \to \infty} \left(a + ar + \cdots + ar^n\right) = \dfrac{a}{1-r}.$

22. *x* is an integer if it is a natural number. (Use set symbols $\mathbb{Z}, \mathbb{N}.$)

23. If $\sum_{n=1}^{\infty} |u_n|$ converges, so does $\sum_{n=1}^{\infty} u_n.$

24. The convergence of $\sum_{n=1}^{\infty} |u_n|$ is sufficient for the convergence of $\sum_{n=1}^{\infty} u_n.$

25. $a \in \mathbb{R}$ is a necessary condition for $a \in \mathbb{Q}.$

26. An integer is a rational number. *Hint:* Use a variable *x* and set symbols; e.g., $\mathbb{Z}, \mathbb{Q}.$

27. Integers are rationals.

28. A necessary condition for two lines in a plane l_1 and l_2 to be parallel is that $l_1 \cap l_2 = \varnothing.$

29. A square is a rectangle.

30. Triangles are polygons.

31. $3x = 3y$ since $x = y$.

32. $f'(x) = 2x$ when $f(x) = x^2$.

33. Squares are not triangles.

1.5 BICONDITIONALS AND COMBINATIONS OF CONNECTIVES

Biconditional. A sentence of the type

$$(P \to Q) \wedge (Q \to P)$$

is called a ***biconditional***, symbolized

$$P \leftrightarrow Q.$$

When P and Q are sentences, the truth table for $P \leftrightarrow Q$ is

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

and is derived from the truth tables for $\to$ and $\wedge$ as follows. We first set up all combinations of truth values for P and Q. Then we use these to find truth values for $P \to Q$, $Q \to P$, and finally $(P \to Q) \wedge (Q \to P)$ which is $P \leftrightarrow Q$.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$ or $P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Thus, $P \leftrightarrow Q$ is true when P and Q are both true or both false.

In mathematics $P \leftrightarrow Q$ is encountered in many forms. The following have the same meaning:

$P \leftrightarrow Q$,

P is equivalent to Q,

P is a necessary and sufficient condition for Q,

Q is a necessary and sufficient condition for P,

P if and only if Q,

Q if and only if P,

P iff Q (“iff” is an abbreviation for “if and only if”),

If P, then Q and conversely,

If Q, then P and conversely.

(Memorize these)

For example,

$$5x = 15 \text{ if and only if } x = 3$$

would be translated by

P: $5x = 15$

Q: $x = 3$

to $P \leftrightarrow Q$.

A meaning of $P \leftrightarrow Q$ is

“P is a necessary and sufficient condition for Q.”

This is explained via the definition of $P \leftrightarrow Q$: $(P \to Q) \wedge (Q \to P)$. If P is a necessary condition for Q, then $Q \to P$. If P is a sufficient condition for Q, then $P \to Q$.

Combination of Connectives. Combinations of $\sim$, $\to$, $\leftrightarrow$, $\wedge$, and $\vee$ often occur. A facility at recognizing them is essential for mathematical reading and proof.

EXAMPLE. We could translate

If p is prime, then if p is even p must be smaller than 7

as follows:

P: p is prime,

Q: p is even,

R: p must be smaller than 7.

The translated sentence would be

$$P \to (Q \to R);$$

that is,

P implies that *Q* implies *R*.

EXAMPLE. Translate

“If *a* is perpendicular to *b* and *b* is perpendicular to *c*, then *a* is parallel to *c*.”

Let

P: *a* is perpendicular to *b*,

Q: *b* is perpendicular to *c*,

and

R: *a* is parallel to *c*.

Then the translated sentence is

$$(P \wedge Q) \to R.$$

EXAMPLE. Translate

“If lines *l* and *m* are not parallel, then *l* and *m* intersect.”

Let

P: lines *l* and *m* are parallel,

Q: *l* and *m* intersect,

then the translated sentence is

$$\sim P \to Q$$

or not *P* implies *Q*. You could have let

R: lines *l* and *m* are **not** parallel,

then the translated sentence would have been $R \to Q.$ The first translation is more desirable because it reveals more logical structure. Such structure will become important when we study proof.

Exercise Set 1.5

Find the truth value.

1. $2<1 \leftrightarrow 2<3$

2. π is real ↔ π is irrational

3. 2 is real ↔ 2 is irrational

4. 2 is real ↔ 2 is rational

5. e is rational ↔ e is an integer (Note: $e = 2.718\ldots$; it is a constant symbol rather than a variable.)

Translate to a sentence of the type $P \leftrightarrow Q$.

6. $x = 5$ if and only if $2x = 10$

7. $x \in \mathbb{Q}$ is a necessary and sufficient condition for $x = p/q$ where $p \in \mathbb{Z},\ q \in \mathbb{Z},$ and $q \neq 0$.

8. A necessary and sufficient condition for the sequence $\{x_n\}_{n=1}^{\infty}$ to have a limit is that the absolute value $|x_n - x_m|$ approaches 0 as m and n go to infinity.

9. $ab = 0$ if and only if $a = 0$ or $b = 0$.

10. If a triangle is isosceles, then it must have two sides equal and conversely.

11. $2x-1=0$ is equivalent to $x=1/2$.

12. f is continuous if and only if f is differentiable.

Translate using $\rightarrow,\ \leftrightarrow,\ \sim,\ \wedge,$ and $\vee$.

13. If p and q are integers and $q \neq 0$, then p/q is a rational number.

14. If ABC is a triangle and ABC is isosceles, then ABC has two equal sides.

15. If $a,\ b,\ c,$ and x are real numbers, $a \neq 0,\ ax^2+bx+c=0,$ and $b^2-4ac=0,$ then the roots of $ax^2+bx+c=0$ are real and equal.

16. If a series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

17. If u_n does not approach 0 as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ cannot be convergent.

18. If a is an integer, then a is even or a is odd.

19. f is differentiable and g is differentiable only if $g \circ f$ is differentiable.

20. If u and v are differentiable functions of x, uv is also differentiable and $\frac{d}{dx}(uv) = u\left(\frac{dv}{dx}\right) + v\left(\frac{du}{dx}\right)$.

21. $x \in J$ or $x \in \mathbb{Q}$ is a necessary and sufficient condition for $x \in \mathbb{R}$.

22. $x \in {}^{-}\mathbb{N}$ or $x \in \mathbb{N}_0$ is equivalent to $x \in \mathbb{Z}$.

23. $x \in A \cap B$ iff $(x \in A$ and $x \in B)$

24. $x \in A \cup B$ iff $(x \in A$ or $x \in B)$

25. $x \in A'$ is a necessary and sufficient condition for $x \notin A$.

1.6 QUANTIFIERS

Sentences involving the phrases "For every…" and "There exists…" also play an important role in the structure of mathematical sentences. For example, the sentences

For every x, $x + 0 = x$

and

There exists an x such that $x^2 = 2$

express certain properties of the real number system.

The Universal Quantifier. The symbol $\forall$, called the ***universal quantifier***, symbolizes phrases such as "For each," "For all," and "For every." A sentence such as

For every x, $P(x)$

translates to

$\forall x P(x)$, or $\forall x,\ P(x)$.

The following have the same meaning:

$\forall x,\ x$ is an integer $\to x \in \mathbb{Q}$,

For every x, if x is an integer, then $x \in \mathbb{Q}$,

For all x, if x is an integer, then $x \in \mathbb{Q}$,

For each x, if x is an integer, then $x \in \mathbb{Q}$,

Every integer belongs to $\mathbb{Q}$,

Every integer is a rational number.

In some mathematics books a sentence like

If x is an integer, then $x \in \mathbb{Q}$

is understood to mean

$\forall x,\ x$ is an integer $\to x \in \mathbb{Q}$.

That is, the universal quantifier is understood and not written. This is pointed out to enable you to interpret sentences you read since each author has his own style of writing.

As another example, in trigonometry the sentences

$$\sin^2 u + \cos^2 u \equiv 1$$

and

$$\sin^2 u + \cos^2 u = 1$$

mean

$$\forall u, \sin^2 u + \cos^2 u = 1,$$

where the quantifier refers to the set of real numbers.

The Existential Quantifier. The symbol $\exists$, called the ***existential quantifier***, symbolizes phrases such as "There exists," "There is at least one," "For at least one," and "Some." A sentence such as

There exists an x such that $P(x)$

translates to

$\exists x P(x),$ or $\exists x,\ P(x).$

The following have the same meaning:

$\exists x,\ x$ is a natural number,

There exists an x such that x is a natural number,

Some number is natural,

There is at least one natural number.

It is important to realize that $\exists$ means "There exists **at least one**"; there is nothing to prevent there being more. For example, compare the sentences

$\exists x,\ x = 0$

and

$\exists x,\ \sin x = 1.$

For

$\exists x,\ x = 0,$

we know there is only one x such that $x = 0$, but for

$$\exists x, \ \sin x = 1,$$

we know there is at least one, in fact many numbers x such that $\sin x = 1$.

Combinations of Quantifiers. Quantifiers may appear together. Consider the following examples. The sentence

For every x and for every y, $x + y = 0$

translates to

$$\forall x \forall y, \ x + y = 0.$$

The sentence

For every x there exists a y such that $x + y = 0$

translates to

$$\forall x \exists y, \ x + y = 0.$$

The sentence

There exists an x such that for every y, $x + y = 0$

translates to

$$\exists x \forall y, \ x + y = 0.$$

The sentence

There exists an x and there exists a y such that $x + y = 0$

translates to

$$\exists x \exists y, \ x + y = 0.$$

Quantifiers may not appear together. For example, the sentence

For every x, if is even, then there exists a y such that $x = 2y$

translates to

$$\forall x \left(x \text{ is even} \rightarrow \exists y, x = 2y \right).$$

Truth Values of Quantified Sentences. Quantifiers refer to a universal set. Sometimes the universal set is pointed out, but sometimes it must be inferred from context. For example, consider the sentence

$\forall x,\ x$ is a triangle $\rightarrow x$ is a polygon.

The universal set might be the set of figures in the plane, or could be the power set of points in the plane. In calculus, the quantifiers usually refer to such universal sets as the set of real numbers, the set of positive real numbers, or the set of real functions.

Henceforth, we consider only nonempty universal sets. (The empty set is not very interesting to study.)

Definition. a) The sentence $\forall x P(x)$ is **true** iff the solution set of $P(x)$ equals the universal set (or, for every replacement of x by a member u of the universal set, $P(u)$ is true).

b) The sentence $\forall x P(x)$ is **false** iff the solution set of $P(x)$ does not equal the universal set (or, there exists a replacement u in the universal set such that $P(u)$ is false).

EXAMPLES.

Sentence $\forall x P(x)$	Universal Set	Solution Set of $P(x)$	Truth Value
$\forall x,\ x=0$	$\{0\}$	$\{0\}$	T
$\forall x,\ x=0$	$\{0,1\}$	$\{0\}$	F
$\forall x,\ x<x+1$	$\mathbb{R}$	$\mathbb{R}$	T
$\forall x,\ 2x^2+3x+1=0$	$\mathbb{N}$	$\varnothing$	F
$\forall x,\ 2x^2+3x+1=0$	$^{-}\mathbb{N}$	$\{-1\}$	F

Definition. a) The sentence $\exists x P(x)$ is **true** iff the solution set of $P(x)$ is nonempty (or, there exists a replacement u such that $P(u)$ is true).

b) The sentence $\exists x P(x)$ is **false** iff the solution set of $P(x)$ is empty (or, for every replacement of x by a member u of the universal set, $P(u)$ is false).

EXAMPLES.

Sentence $\exists x P(x)$	Universal Set	Solution Set of $P(x)$	Truth Value
$\exists x,\ x = 0$	$\{0\}$	$\{0\}$	T
$\exists x,\ x = 0$	$\{0,1\}$	$\{0\}$	T
$\exists x,\ x < x+1$	$\mathbb{R}$	$\mathbb{R}$	T
$\exists x,\ 2x^2+3x+1=0$	$\mathbb{N}$	$\varnothing$	F
$\exists x,\ 2x^2+3x+1=0$	$^{-}\mathbb{N}$	$\{-1\}$	T

Exercise Set 1.6

Translate to logical symbolism.

1. Every triangle is a polygon.
2. For every x, if x is a natural number, then x is an integer.
3. For each natural number x, x is even or x is odd.
4. There exists an x such that x is prime and x is even.
5. There is an x such that $x = \lim_{n \to \infty} \frac{1}{n}$.
6. There is an X such that $n \le X \le 2$ and $\int_n^2 f(x)dx = (2-n)f(X)$.
7. There exists a p and there exists a q such that $p \cdot q = 32$.
8. For every x there exists a y such that $x < y$.
9. There exists a y such that for every x, $x + 0 = y$.
10. There exists an x and there exists a y such that x^y is irrational.
11. For every x and for every y, $x + y = y + x$.
12. There exist an x and y such that $x^2 = y$.
13. For every x and y, $xy = yx$.
14. $1 < 2 \rightarrow$ there exists an x such that $x < 2$.
15. For every x, $\sqrt{x} = k$ implies that there is a y such that $\sqrt{y} = k$.

Translate using $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\exists$, $\forall$, and the following symbols for sentences.

$D(f)$: f is differentiable

$C(f)$: f is continuous

$S(x)$: x is a square

$R(x)$: x is a rectangle

$E(x)$: x is an equilateral triangle

$A(x)$: x is an equiangular triangle

$V(x)$: x is even

$O(x)$: x is odd

$L(x)$: x is isosceles

16. Every differentiable function is continuous.

17. There are continuous functions which are not differentiable.

18. All equilateral triangles are equiangular and some equiangular triangles are equilateral.

19. If every number which is even is not odd, then some odd numbers cannot be even.

20. If all equilateral triangles are isosceles, then some isosceles triangles are not equilateral.

21. All squares are rectangles.

For each sentence of the type $\forall xP(x)$ and indicated universal set find the solution set of $P(x)$, then the truth value of $\forall xP(x)$.

	Sentence $\forall xP(x)$	Universal Set	Solution Set of $P(x)$	Truth Value
22.	$\forall x,\ x<0$	$\mathbb{N}$		
23.	$\forall x,\ x<0$	$\mathbb{Z}$		
24.	$\forall x,\ x<0$	$^{-}\mathbb{N}$		
25.	$\forall x,\ x^2+2x+1=0$	$\mathbb{R}$		
26.	$\forall x,\ x+0=0+x=x$	$\mathbb{R}$		
27.	$\forall x,\ x^2-1=(x-1)(x+1)$	$\mathbb{R}$		

For each sentence of the type $\exists xP(x)$ and indicated universal set find the solution set of $P(x)$, then the truth value of $\exists xP(x)$.

	Sentence $\exists x P(x)$	Universal Set	Solution Set of $P(x)$	Truth Value
28.	$\exists x,\ x < 0$	$\mathbb{N}$		
29.	$\exists x,\ x < 0$	$\mathbb{Z}$		
30.	$\exists x,\ x < 0$	$^{-}\mathbb{N}$		
31.	$\exists x,\ x^2 + 2x + 1 = 0$	$\mathbb{R}$		
32.	$\exists x,\ x + 0 = 0 + x = x$	$\mathbb{R}$		
33.	$\exists x,\ x^2 - 1 = (x-1)(x+1)$	$\mathbb{R}$		

34. Compare truth values of the sentences $\forall x P(x)$ and $\exists x P(x)$ encountered in the previous examples and exercises. What can you conjecture about the truth value of $\exists x P(x)$ when $\forall x P(x)$ is true?

1.7 TRUTH VALUES OF MORE COMPLICATED QUANTIFIED SENTENCES

The sentence $\forall x \forall y P(x, y)$. Suppose $P(x, y)$ is a sentence with two variables x and y. The sentence

$$\forall x \forall y P\left(x, y\right)$$

is true iff for every replacement of x and y by members a and b of the universal set,

$P\left(a, b\right)$ is true.

EXAMPLE. The sentence

$$\forall x \forall y,\ x + y = y + x$$

with universal set {0, 1, 2} is true. Note that each of the following is true:

$0 + 1 = 1 + 0$ $\quad$ $1 + 2 = 2 + 1$ $\quad$ $2 + 2 = 2 + 2$

$1 + 0 = 0 + 1$ $\quad$ $2 + 1 = 1 + 2$ $\quad$ $2 + 0 = 0 + 2$

$0 + 2 = 2 + 0$ $\quad$ $1 + 1 = 1 + 1$ $\quad$ $0 + 0 = 0 + 0$

EXAMPLE. The sentence

$$\forall x \forall y,\ y < x$$

with universal set {0, 1, 2} is false. When y is replaced by 2 and x by 1 the sentence

$$2 < 1$$

is false.

The sentence $\exists x \exists y P\left(x, y\right)$. The sentence

$$\exists x \exists y P\left(x, y\right)$$

is true iff there is at least one replacement b for x and at least one replacement c for y such that

$P(b, c)$ is true.

EXAMPLE. The sentence

$$\exists x \exists y,\ x + 3 = 2 \cdot y$$

with universal set $\mathbb{Z}$ is true. When x is replaced by 5 and y is replaced by 4, the sentence

$$5+3=2\cdot 4$$

is true.

EXAMPLE. The sentence

$$\exists x \exists y,\ \ x/y=\sqrt{2}$$

with universal set $\mathbb{Z}$ is false. There are no replacements of *x* and *y* by integers *b* and *c* which make the sentence

$$b/c=\sqrt{2}$$

true. We prove this later.

The sentence $\forall x \exists y P(x,y)$. The sentence

$$\forall x \exists y P(x,y)$$

is true iff for every replacement of *x* by a member of the universal set *b*,

$$\exists y P(b,y)$$

is true.

EXAMPLE. The sentence

$$\forall x \exists y,\ x+y=0$$

with universal set $\{-1,0,1\}$ is true. Note:

$\exists y,\ 0+y=0$ (True; $y=0$)
$\exists y,\ 1+y=0$ (True; $y=-1$)
$\exists y,\ -1+y=0$ (True; $y=1$)

EXAMPLE. The sentence

$$\forall x \exists y,\ y<x$$

with universal set {0, 1, 2} is false. Note:

$\exists y,\ y<0$ (False; solution set of $y<0$ is $\varnothing$)
$\exists y,\ y<1$ (True; $y=0$)
$\exists y,\ y<2$ (True; $y=0$ or 1)

The sentence $\exists y \forall x P(x,y)$. The sentence

$$\exists y \forall x P(x,y)$$

is true iff there exists a replacement c for y such that

$$\forall x P(x,c)$$

is true. Thus the same c makes the sentence

$$P(b, c)$$

true for every element b in the universal set.

EXAMPLE. The sentence

$$\exists y \forall x,\ x + y = x$$

with universal set {0, 1, 2} is true because the sentence

$$\forall x,\ x + 0 = x$$

is true.

EXAMPLE. The sentence

$$\exists y \forall x,\ y > x$$

with universal set {0, 1, 2} is false because each of the sentences

$$\forall x,\ 0 > x$$

$$\forall x,\ 1 > x$$

$$\forall x,\ 2 > x$$

is false; that is, there is no replacement b for y which makes the sentence

$$\forall x,\ b > x$$

true.

There is an important distinction between the sentences

$$\forall x \exists y P(x,y)$$

and

$$\exists y \forall x P(x, y).$$

If the sentence

$$\forall x \exists y P(x, y) \text{ is true}$$

there is a dependence asserted between y and x. That is, the y depends on the x. If the sentence

$$\exists y \forall x P(x, y) \text{ is true}$$

there is no dependence of y on x. The same y makes $P(x, y)$ true for all x.

Later we will prove that every sentence of the type

$$\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$$

is true.

Exercise Set 1.7

Decide the truth value of each sentence with indicated universal set.

1. The universal set is {0, 1, 2}.

   a) $2 < 1 \rightarrow \exists x,\ x < 0$

   b) $\forall x \exists y,\ y < x$

   c) $\exists y \forall x,\ y < x$

   d) $\exists y \forall x,\ y < x + 1$

   e) $\forall x \exists y,\ y \leq x$

   f) $\exists y \forall x,\ y \leq x$

   g) $\forall x \exists y,\ x + y = 0$

   h) $\forall x \forall y,\ x + y = y + x$

   i) $\exists x \exists y,\ x + 5 = 2 + y$

2. The universal set is $\mathbb{N}$. Answer (a) through (i) above. Compare your answers to those of Exercise 1.

3. The universal set is $\mathbb{Z}$. Answer (a) through (i) above. Compare your answers to those of Exercises 1 and 2.

4. The universal set is the set of all real functions.

a) $\forall f\,(f \text{ is differentiable})$

b) $\forall f\,(f \text{ is differentiable} \to f \text{ is continuous})$

c) $\exists f\,(f \text{ is continuous} \wedge f \text{ is differentiable})$

d) $\exists f\,(f \text{ is continuous} \wedge f \text{ is not differentiable})$

5. The universal set is the set of all infinite sequences $\{u_n\}$ of real numbers.

a) $\forall \{u_n\}, \left(\sum |u_n| \text{ is convergent} \to \sum u_n \text{ is convergent}\right)$

b) $\forall \{u_n\}, \left(\{u_n\} \text{ is convergent} \vee \{u_n\} \text{ is divergent}\right)$

c) $\exists \{u_n\}, \left(\lim\limits_{n\to\infty} u_n = 0 \wedge \sum u_n \text{does not converge}\right)$. Explain.

6. Consider the sentence $x < y$ with universal set $\mathbb{Z}$.

a) Decide the truth value of $\forall x \exists y,\ x < y$.

b) Decide the truth value of $\exists y \forall x,\ x < y$.

c) Decide the truth value of $\forall x \exists y,\ x < y \to \exists y \forall x,\ x < y$.

d) Is every sentence of the type $\forall x \exists y P(x, y) \to \exists y \forall x P(x, y)$ true? Why?

e) Decide the truth value of $\exists y \forall x,\ x < y \to \forall x \exists y,\ x < y$.

f) Use truth values for $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$ in universal sets previously considered to compute truth values for $\exists y \forall x P(x, y) \to \forall x \exists y P(x, y)$. Does it seem that every sentence of the type $\exists y \forall x P(x, y) \to \forall x \exists y P(x, y)$ is true?

7. Read the excellent article: E. A. Kuehls, "The Truth-Value of $\{\forall, \exists, P(x, y)\}$: A Graphical Approach," *Mathematics Magazine*, Vol. 43, Nov. 1970, p. 260.

8. Let $P(x)$: x is irrational

$Q(x)$: x is rational

Decide the truth value of each of the following with universal set $\mathbb{R}$.

a) $\forall x\left[P(x) \vee Q(x)\right]$

b) $\forall x P(x)$

c) $\forall x Q(x)$

d) $\forall x P(x) \vee \forall x Q(x)$

e) $\forall x\left[P(x) \vee Q(x)\right] \rightarrow \left[\forall x P(x) \vee \forall x Q(x)\right]$

f) Is every sentence of the type $\forall x\left[P(x) \vee Q(x)\right] \rightarrow \left[\forall x P(x) \vee \forall x Q(x)\right]$ true?

9. By a procedure similar to Exercise 8 decide if every sentence of the type $\left[\exists x P(x) \wedge \exists x Q(x)\right] \rightarrow \exists x\left[P(x) \wedge Q(x)\right]$ is true.

1.8 REASONING SENTENCES

Mathematicians assume a certain class of sentences to be true before they ever prove any theorems in a mathematical system. We call these **reasoning sentences** or **rules of reasoning**. These rules are assumed by the mathematician, and accordingly could be called reasoning axioms.

Tautologies. An important class of these reasoning sentences are known as tautologies. A ***tautology*** is a sentence which is true no matter what the truth value of its constituent parts. (A review of truth tables for $\wedge$, $\vee$, $\sim$, $\rightarrow$, $\leftrightarrow$ would be helpful at this time.)

EXAMPLE. The sentence

$$P \rightarrow (P \vee Q)$$

is a tautology, where P and Q represent arbitrary mathematical sentences. We show this with a truth table. Truth values are obtained by successively breaking the sentence up into its constituent parts and computing truth values. Hence

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Note that the truth values for $P \vee Q$ were determined first and listed in column three. Then columns one and three were used to determine column four.

EXAMPLE. Show $(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$ is a tautology.

P	Q	$\sim P$	$\sim Q$	$P \rightarrow Q$	$\sim Q \rightarrow \sim P$	$(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Related to every conditional

$$P \to Q$$

is another conditional

$$\sim Q \to \sim P$$

called its ***contrapositive***. We have just shown that the two are equivalent; that is, $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$.

Exercise Set 1.8

Use truth tables to determine which of the following are tautologies.

1. $[P \wedge (P \to Q)] \to Q$ (*Modus ponens*)

2. $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ (*Law of Syllogism*; this truth table requires 8 different combinations of truth values at the outset)

$$\left.\begin{array}{l} 3.\ \sim(P \wedge Q) \leftrightarrow (\sim P \vee \sim Q) \\ 4.\ \sim(P \vee Q) \leftrightarrow (\sim P \wedge \sim Q) \end{array}\right\} \quad (\textit{De Morgan's Law})$$

5. $\sim(P \to Q) \leftrightarrow (P \wedge \sim Q)$

6. $(P \to Q) \leftrightarrow (\sim P \vee Q)$

7. $(P \wedge Q) \to P$

8. $\sim \sim P \leftrightarrow P$

9. $(P \wedge Q) \to (P \vee Q)$

10. $(P \to \sim Q) \to (Q \to \sim P)$

11. $(P \to Q) \to (Q \to P)$

12. $(P \vee Q) \to (P \wedge Q)$

13. $\sim P \to P$

14. $\left[\left(P\wedge R\right)\leftrightarrow\left(P\wedge Q\right)\right]\rightarrow\left(R\leftrightarrow Q\right)$

$$\left.\begin{array}{ll} 15. & \left[\sim P\rightarrow\left(R\wedge\sim R\right)\right]\rightarrow P \\ 16. & \left[\left(P\wedge\sim Q\right)\wedge\left(R\wedge\sim R\right)\right]\rightarrow\left(P\rightarrow Q\right) \end{array}\right\}\quad(\textit{Proof by Contradiction})$$

17. $P\vee\sim P$

1.9 VALID ARGUMENTS

More Tautologies. The tautologies in the preceding exercise set are quite useful. Below are several other useful tautologies.

$$P \leftrightarrow P$$

$$P \rightarrow P$$

$$\left[P \rightarrow (Q \vee R)\right] \rightarrow \left[(P \wedge \sim Q) \rightarrow R\right]$$

$$\left[(P \rightarrow S_1) \wedge (S_1 \rightarrow S_2) \wedge \cdots \wedge (S_{n-1} \rightarrow S_n) \wedge (S_n \rightarrow R)\right] \rightarrow [P \rightarrow R] \qquad \text{(Law of Syllogism)}$$

$$\left[(P \rightarrow R) \wedge (Q \rightarrow R)\right] \rightarrow \left[(P \vee Q) \rightarrow R\right] \qquad \text{(Proof by Cases)}$$

$$\left.\begin{aligned} &(P \wedge Q) \leftrightarrow (Q \wedge P) \\ &(P \vee Q) \leftrightarrow (Q \vee P) \end{aligned}\right\} \quad \text{(Commutative Laws)}$$

$$\left[P \rightarrow (R \rightarrow Q)\right] \leftrightarrow \left[(P \wedge R) \rightarrow Q\right]$$

$$\left.\begin{aligned} &\left[P \wedge (Q \wedge R)\right] \leftrightarrow \left[(P \wedge Q) \wedge R\right] \\ &\left[P \vee (Q \vee R)\right] \leftrightarrow \left[(P \vee Q) \vee R\right] \end{aligned}\right\} \quad \text{(Associative Laws)}$$

$$\left.\begin{aligned} &\left[P \wedge (Q \vee R)\right] \leftrightarrow \left[(P \wedge Q) \vee (P \wedge R)\right] \\ &\left[P \vee (Q \wedge R)\right] \leftrightarrow \left[(P \vee Q) \wedge (P \vee R)\right] \end{aligned}\right\} \quad \text{(Distributive Laws)}$$

$$\left[(P \leftrightarrow Q_1) \wedge \cdots \wedge (Q_n \leftrightarrow Q)\right] \rightarrow (P \leftrightarrow Q)$$

LOGICAL AXIOM 1. Every tautology is a rule of reasoning.

The preceding tautologies are not all there are. If you want to make a deduction based on a sentence, check its truth table. If it is a tautology, use it. Tautologies provide lots of reasoning theorems before we ever start deduction within a mathematical system.

There are actually two branches of formal logic: the *statement calculus*, involving statements and reasoning by tautology, and the *predicate calculus*, involving quantified sentences. In this text we are studying logic informally, with the goal being to give you a working knowledge of logic. Thus, we will not go into these branches in great detail. From the predicate calculus we get another collection of reasoning sentences, some of which are listed in Logical Axiom 2. These cannot be verified by tautology.

LOGICAL AXIOM 2. Let U be a universal set. Each of the following is a rule of reasoning:

(1) $\left[\forall x,\ P(x) \rightarrow Q(x)\right] \rightarrow \left[\forall x P(x) \rightarrow \forall x Q(x)\right],$

(2) $\forall x P(x) \leftrightarrow \left[P(u),\ \text{for any } u \in U\right],$

(3) $\exists x P(x) \leftrightarrow \left[P(u),\ \text{for some } u \in U\right].$

An ***argument*** is an assertion that from a certain set of sentences $S_1, \ldots, S_n$ (called ***premises*** or ***assumptions***) one can deduce another sentence Q (called a ***conclusion*** or ***inference***). Such an argument will be denoted

$$S_1, \ldots, S_n \vdash Q$$

Arguments are either ***valid*** (correct) or ***invalid*** (incorrect).

Definition. $S_1, \ldots, S_n \vdash Q$ is a valid argument iff $(S_1 \wedge \cdots \wedge S_n) \rightarrow Q$ is a rule of reasoning.

Rule of Substitution. Suppose $P \leftrightarrow Q$. Then P and Q may be substituted for one another in any sentence.

EXAMPLE.

$$\begin{array}{l} P \leftrightarrow Q \\ \underline{R \rightarrow (S \wedge Q)} \\ \therefore R \rightarrow (S \wedge P) \end{array}$$

You may insert a tautology into any set of premises.

EXAMPLE.

$$\begin{array}{ll} P \rightarrow Q & \text{Premise} \\ (P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P) & \text{Tautology} \\ \hline \therefore \sim Q \rightarrow \sim P & \end{array}$$

EXAMPLE. $P,\ P \rightarrow Q \vdash Q$ is valid because $[P \wedge (P \rightarrow Q)] \rightarrow Q$ is a tautology.

EXAMPLE. $Q,\ P \rightarrow Q \vdash P$ is *not* a valid argument because $[Q \wedge (P \rightarrow Q)] \rightarrow P$ is not a tautology. (Such an inference might be referred to as “modus humorous.”)

EXAMPLE. $\forall x P(x) \vdash P(u)$ for any $u \in U$, a universal set, is a valid argument by part (2) of Logical Axiom 2.

Some types of valid arguments are used so much we give them special names.

Rule of Modus Ponens. From any conditional $P \rightarrow Q$ and P, one may conclude Q; that is, P, $P \rightarrow Q$, therefore Q. This is a valid argument and can be denoted two ways.

a) $P, P \rightarrow Q \vdash Q$

b)
$$\begin{array}{l} P \to Q \\ \underline{\quad P \qquad\qquad} \\ \therefore Q \end{array}$$

This rule is based on the tautology

$$\left[P \wedge (P \to Q)\right] \to Q.$$

When using modus ponens the form of the sentence is important.

$$\begin{array}{ll} 1. & (\qquad) \to [\qquad] \\ \underline{2.} & \underline{(\qquad)\qquad\qquad\quad} \\ 3. & \therefore\ [\qquad] \end{array}$$

Having placed sentences in the parentheses (the same sentences must be in both sets of parentheses) and the brackets, and assuming sentences 1 and 2 true, we deduce sentence 3.

EXAMPLE. If $f(x) = \sin x,$ then $f'(x) = \cos x$

$$\begin{array}{l} \underline{f(x) = \sin x \qquad\qquad\qquad\qquad} \\ \therefore\ f'(x) = \cos x \end{array}$$

EXAMPLE. $x = 5$ only if $2x = 10$

$$\begin{array}{l} \underline{x = 5 \qquad\qquad\qquad} \\ \therefore 2x = 10 \end{array}$$

EXAMPLE. Deduce $\sim P$ from the premises $\sim Q$ and $P \to Q.$

$$\begin{array}{ll} & \left.\begin{array}{l} 1.\ \ \sim Q \\ 2.\ \ P \to Q \end{array}\right\} \text{Premises} \\ & \underline{3.\ \ (P \to Q) \to (\sim Q \to \sim P) \qquad \text{Tautology}} \\ \therefore & 4.\ \ \sim Q \to \sim P, \text{ by modus ponens on 2 and 3} \end{array}$$

$$\begin{array}{ll} & 1.\ \ \sim Q \\ & \underline{4.\ \ \sim Q \to \sim P \qquad\qquad\qquad} \\ \therefore & 5.\ \ \sim P, \text{ by modus ponens} \end{array}$$

EXAMPLE. Deduce S from the premises $\sim S \to H$, $H \to \sim I$, and I.

$$\left.\begin{array}{l} 1.\ \ \sim S \to H \\ 2.\ \ H \to \sim I \\ 3.\ \ I \end{array}\right\}\quad \text{Premises}$$

$$\begin{array}{ll} 4.\ \ (H \to \sim I) \to (I \to \sim H) & \text{Tautology} \\ \hline \therefore 5.\ \ I \to \sim H,\ \text{by modus ponens on 2 and 4} & \end{array}$$

$$\begin{array}{l} 5.\ \ I \to \sim H \\ 3.\ \ I \\ \hline \therefore 6.\ \ \sim H, \text{by modus ponens} \end{array}$$

$$\begin{array}{ll} 1.\ \ \sim S \to H & \\ 7.\ \ (\sim S \to H) \to (\sim H \to S) & \text{Tautology} \\ \hline \therefore 8.\ \ \sim H \to S,\ \text{by modus ponens} & \end{array}$$

$$\begin{array}{l} 8.\ \ \sim H \to S \\ 6.\ \ \sim H \\ \hline \therefore S,\ \text{by modus ponens} \end{array}$$

Exercise Set 1.9

Complete.

1. $3x = 15$ if $x = 5$

$$\begin{array}{l} \underline{x = 5 \qquad\qquad} \\ \therefore \underline{\qquad\qquad} \end{array}$$

2. $P,\ P \leftrightarrow Q \vdash$ ____________

3.
$$\begin{array}{ll} P & \text{Premise} \\ P \to (\underline{\qquad\quad}) & \text{Tautology} \\ \hline \therefore P \vee Q & \end{array}$$

Determine the validity.

4. $\sim (P \to Q) \vdash P$

5. $\sim P \wedge \sim Q \vdash \sim P$

6. $P \to (Q \vee R), P \wedge \sim Q \vdash R$

7. $(P \wedge R) \to Q, P \wedge R \vdash Q$

8. $\sim P, P \vee Q \vdash Q$

9. $P \to R, Q \to P, \sim R \vdash \sim Q$

10. $P \to Q, R \to \sim Q \vdash R \to \sim P$

11. $P \vee Q \vdash Q$

12. $P \to Q \vdash Q \to P$ ($Q \to P$ is the ***converse*** of $P \to Q$)

13. $P \wedge Q \vdash P \vee Q$

14. $P \vee Q \vdash P \wedge Q$

15. $\exists x P(x) \vdash \mathrm{P}(u)$ for some $u \in U$. U is a universal set.

16. $[\forall x P(x) \to Q(x)] \vdash \forall x P(x) \to \forall x Q(x)$

1.10 CONTRAPOSITIVES

Recall that the contrapositive of $P \to Q$ is $\sim Q \to \sim P$. The two sentences are equivalent. Contrapositives reveal added insight.

EXAMPLE. In calculus the sentence

$$\text{If } \sum_{n=1}^{\infty} u_n \text{ converges, then } \lim_{n\to\infty} u_n = 0$$

is true. If we translate the sentence

$$P\text{: } \sum_{n=1}^{\infty} u_n \text{ converges}$$

$$Q\text{: } \lim_{n\to\infty} u_n = 0$$

then $P \to Q$ is true. Its contrapositive is also true because $(P \to Q) \leftrightarrow (\sim Q \to \sim P)$ is a tautology and, by the rule of substitution, $\sim Q \to \sim P$ is true. That is,

$$\text{If } \lim_{n\to\infty} \neq 0, \text{ then } \sum_{n=1}^{\infty} u_n \text{ diverges}$$

is true. Recall that to check the convergence of a series you determine if the *n*th term converges to zero. If it does not, you know the series cannot converge. The contrapositive justifies this.

EXAMPLE. In the system of integers the sentence

x odd $\to$ x^2 odd

is true (for any given x). Thus the contrapositive

x^2 even (not odd) $\to$ x even (not odd)

is true.

Exercise Set 1.10

1. Form the contrapositive of the sentence "If $f(x) = \sin x$, then $f'(x) = \cos x$." Is this new sentence true? Why?

Form the contrapositive of the following. Use →.

2. x is odd is a necessary condition for x not being even.

3. x is rational is a sufficient condition for x to be real.

4. If f is differentiable, then f is continuous.

5. $x \in A \to x \in B$

6. x^2 odd → x odd

7. x^2 even → x even

8. x is even only if x^2 is even

9. $A \cap B \neq \varnothing \to A \neq \varnothing$

10. $\left[\forall \varepsilon > 0 \left(|x| < \varepsilon\right)\right] \to x = 0$

11. $f(x) = f(y) \to x = y$

12. $x \leq y \to f(x) \leq f(y)$

13. $x < y \to f(x) < f(y)$

14. $\sum_{n=1}^{\infty} |u_n|$ converges $\to \sum_{n=1}^{\infty} u_n$ converges

1.11 NEGATIONS

Negations of $\forall x P(x)$ and $\exists x P(x)$. It is often useful to express the negations of sentences of the type $\forall x P(x)$ and $\exists x P(x)$ in other forms.

THEOREM 1. Every sentence of the type

$$\sim \forall x P(x) \leftrightarrow \exists x \sim P(x)$$

is true.

Proof. We prove that $\sim \forall x P(x)$ and $\exists x \sim P(x)$ are equivalent by showing that their truth values agree.

Suppose $\sim \forall x P(x)$ is true. Then $\forall x P(x)$ is false; so there exists a replacement u in the universal set such that $P(u)$ is false. Then $\sim P(u)$ is true for this replacement u. Thus,

$$\exists x \sim P(x) \text{ is true.}$$

Suppose $\sim \forall x P(x)$ is false. Then $\forall x P(x)$ is true, so for every replacement u $P(u)$ is true. Hence, for every replacement u, $\sim P(u)$ is false. Thus

$$\exists x \sim P(x) \text{ is false.}$$

Therefore, $\sim \forall x P(x)$ and $\exists x \sim P(x)$ are equivalent. □

(Henceforth, □ will denote the completion of a proof.)

Applying the tautology $(P \leftrightarrow Q) \leftrightarrow (\sim P \leftrightarrow \sim Q)$ to the previous theorem we get

$$\forall x P(x) \leftrightarrow \sim \exists x \sim P(x).$$

We have proved the following theorem.

THEOREM 2. Every sentence of the type

$$\forall x P(x) \leftrightarrow \sim \exists x \sim P(x)$$

is true.

Using Theorems 1 and 2 we can prove the following theorems.

THEOREM 3. Every sentence of the type

$$\sim \exists x P(x) \leftrightarrow \forall x \sim P(x)$$

is true.

Proof. By Theorem 2, $\sim \exists x \sim\sim P(x) \leftrightarrow \forall x \sim P(x),$ where we substitute $\sim P(x)$ for $P(x).$ Now by a tautology we know $\sim\sim P(x) \leftrightarrow P(x).$ Hence, $\sim \exists x \sim\sim P(x) \leftrightarrow \sim \exists x P(x).$ Thus we have $\sim \exists x P(x) \leftrightarrow \forall x \sim P(x)$ by the Rule of Substitution. □

THEOREM 4. Every sentence of the type

$$\exists x P(x) \leftrightarrow \sim \forall x \sim P(x)$$

is true.

Proof. Left as an exercise.

We have established the rules:

1) $\sim \forall x P(x) \leftrightarrow \exists x \sim P(x)$

2) $\forall x P(x) \leftrightarrow \sim \exists x \sim P(x)$

3) $\sim \exists x P(x) \leftrightarrow \forall x \sim P(x)$

4) $\exists x P(x) \leftrightarrow \sim \forall x \sim P(x)$

The usefulness of these rules will be realized when doing proofs by contrapositive and by contradiction.

Simplified Negations. Moving the negation symbol past the quantifiers of a sentence provides a more meaningful, simplified translation of the negation.

EXAMPLE. A simplified negation of

$$\exists y \forall x,\ xy \leq 2$$

is

$$\forall y \exists x,\ xy > 2.$$

To show this notice that

$$\begin{aligned} \sim \exists y \forall x,\ xy \le 2 &\leftrightarrow \forall y \sim \forall x,\ xy \le 2,\ \text{by (3)} \\ &\leftrightarrow \forall y \exists x, \sim \left(xy \le 2\right),\ \text{by (1)} \\ &\leftrightarrow \forall \text{y} \exists \text{x},\ xy > 2,\ \text{by substituting } xy > 2 \text{ for } \sim \left(xy \le 2\right) \end{aligned}$$

Perhaps you have discovered that forming a simplified negation which begins with a series of quantifiers amounts to changing each existential quantifier to a universal quantifier and vice versa and moving the negation symbol to the right of the quantifiers.

EXAMPLE. A simplified negation of

$$\forall x \exists y \forall z,\ xy = z$$

is

$$\exists x \forall y \exists z,\ xy \ne z.$$

Further simplifications of negations can be formed using tautologies.

EXAMPLE. A simplified negation of

$$\exists x\left[P(x) \to Q(x)\right]$$

is

$$\forall x\left[P(x) \wedge \sim Q(x)\right].$$

This is shown as follows.

$$\begin{aligned} \sim \exists x\left[P(x) \to Q(x)\right] &\leftrightarrow \forall x \sim \left[P(x) \to Q(x)\right],\ \text{by (3)} \\ &\leftrightarrow \forall x\left[P(x) \wedge \sim Q(x)\right],\ \text{by the tautology } \sim \left(P \to Q\right) \leftrightarrow \left(P \wedge \sim Q\right) \end{aligned}$$

EXAMPLE. A simplified negation of

$$\forall x \exists y\left[P(x) \wedge y \le x\right]$$

is

$$\exists x \forall y\left[\sim P(x) \vee y > x\right].$$

This is shown as follows.

$$\begin{aligned} \sim \forall x \exists y\left[P(x) \wedge y \le x\right] &\leftrightarrow \exists x \forall y \sim \left[P(x) \wedge y \le x\right] \\ &\leftrightarrow \exists x \forall y\left[\sim P(x) \vee y > x\right],\ \text{by the tautology } \sim \left(P \wedge Q\right) \leftrightarrow \left(\sim P \vee \sim Q\right) \end{aligned}$$

More on the Utility of Negations. Quantifiers within the same sentence can refer to different universal sets. The universal sets can be described within the sentence; for example,

$$\underbrace{\forall n \in \mathbb{N}}_{U=\mathbb{N}} \quad \underbrace{\exists x > 0,}_{U=\text{ positive real nos.}} \quad n^2 > x$$

The rules for negation still hold; that is,

$$\sim \forall n \in \mathbb{N}\, \exists x > 0, n^2 > x \leftrightarrow \exists n \in \mathbb{N}\, \forall x > 0, n^2 \leq x.$$

Suppose we have a complicated sentence as in the following definition.

Definition. *A* is the ***limit*** of the sequence $\{a_n\}_{n=1}^{\infty}$ iff for each $\varepsilon > 0$ there exists a natural number *m* such that for every *n* > *m*, $|a_n - A| < \varepsilon$.

EXAMPLE.

a) Translate the previous definition to logical symbolism.

A is the ***limit*** of $\{a_n\}_{n=1}^{\infty} \leftrightarrow \forall \varepsilon > 0\, \exists m \in \mathbb{N}\, \forall n > m, |a_n - A| < \varepsilon.$

b) Form a simplified negation of the sentence.

A is *not* the ***limit*** of $\{a_n\}_{n=1}^{\infty} \leftrightarrow \exists \varepsilon > 0\, \forall m \in \mathbb{N}\, \exists n > m, |a_n - A| \geq \varepsilon$.

Now if we wanted to prove that *A* is not the limit of $\{a_n\}_{n=1}^{\infty}$ we know what we must show.

A knowledge of logic helped in three ways:

1) It helped translate a complicated sentence into more meaningful symbolism.

2) It enabled us to find a negation of the sentence.

3) With this negation, we knew what had to be shown to prove the first sentence false.

As another example consider the definition of an increasing function.

Definition. A function *f* is ***increasing*** iff for every *x* and for every *y*, if $x \leq y$, then $f(x) \leq f(y)$.

EXAMPLE.

a) Translate the previous definition to logical symbolism.

A function f is ***increasing*** $\leftrightarrow \forall x \forall y \left[x \le y \rightarrow f(x) \le f(y)\right].$

b) Form a simplified negation of the sentence.

A function f is ***not increasing*** $\leftrightarrow \exists x \exists y \left[x \le y \wedge f(x) > f(y)\right].$

EXAMPLE. The function $f(x) = x^2$ is not increasing. When $x = -2$ and $y = 1$,

$$x \le y,$$

$$f(x) = 4,$$

$$f(y) = 1,$$

and

$$f(x) > f(y).$$

Counterexamples. To prove a sentence of the type

$$\forall x P(x)$$

false, one could try to prove

$$\exists x \sim P(x)$$

true. This is referred to as “providing a counterexample.” Thus, the function $f(x) = x^2$ of the previous example is a counterexample to the sentence

Every function is increasing.

Exercise Set 1.11

Form a simplified negation.

1. $\exists x,\ x < 0 \wedge Q(x)$
2. $\forall x \exists y \forall z \forall q \exists j,\ x + y + z + q + j = 0$
3. $\left(P \wedge \sim Q\right) \rightarrow \sim R$

4. $\sim \forall x \forall y \exists z,\ xz = y$

5. $\forall \varepsilon \exists \delta \forall x \left[|x - c| < \delta \rightarrow | f(x) - f(c) | < \varepsilon \right]$

6. $\forall \varepsilon \exists n \forall m \left[m > n \rightarrow \left| a_m - a \right| < \varepsilon \right]$

7. $\left(x \in \mathbb{Q} \wedge y \in J \right) \rightarrow \left(x + y \right) \in J$

8. $\left(P \wedge Q \right) \rightarrow R$

9. $x \in A \wedge x \in B$

10. $x \in A \vee x \in B$

11. $\left(P_1 \wedge \cdots \wedge P_n \right) \rightarrow Q$

12. $P \leftrightarrow Q$

13. **The Archimedean Property.** For every two positive real numbers a and b there exists an $n \in \mathbb{N}$ such that $na > b$.

For each definition in Exercises 14-28

a) express the defining sentence on the right in logical symbolism;

b) express the simplified negation of the defining sentence in logical symbolism.

14. A function f is ***even*** iff for every x, $f(-x) = f(x)$.

15. A function f is ***odd*** iff for every x, $f(-x) = -f(x)$.

16. A function f is ***constant*** iff for every x and for every y, $f(x) = f(y)$.

17. A function f is ***periodic*** iff there exists a $p > 0$ such that for every x, $f(x + p) = f(x)$.

18. A function f is ***decreasing*** iff for every x and for every y, if $x \le y$, then $f(x) \ge f(y)$.

19. A function f is ***strictly increasing*** iff for every x and for every y, if $x < y$, then $f(x) < f(y)$.

20. A function f is ***strictly decreasing*** iff for every x and for every y, if $x < y$, then $f(x) > f(y)$.

21. A function f is ***one-to-one*** iff for every x and for every y, if $f(x) = f(y)$, then $x = y$.

22. A function f from *A* to *B* is ***onto*** iff for every $y \in B$ there exists an $x \in A$ such that $f(x) = y.$

23. A function f has a ***limit*** *L* at x_0 iff for every *x* and for every $\varepsilon > 0,$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta.$

24. A function f is ***bounded*** iff there exists an *M* such that for every *x*, $|f(x)| \le M.$

25. A function f is ***continuous*** at x_0 iff for every *x* and for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta,$ then $|f(x) - f(x_0)| < \varepsilon.$

26. A function f is ***continuous on a set E*** iff for any *x* in *E* and $\varepsilon > 0,$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever *y* is in *E* and $|x - y| < \delta.$

27. A function f is ***uniformly continuous on a set E*** iff for any $\varepsilon > 0,$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever *x* and *y* are in *E* and $|x - y| < \delta.$

28. A sequence $\{a_n\}_{n=1}^{\infty}$ is ***Cauchy*** iff for every $\varepsilon > 0,$ there exists a positive integer n_0 such that $|a_n - a_m| < \varepsilon$ whenever *m* and *n* are greater than $n_0.$

Find counterexamples for each of the following.

29. $\forall \{u_n\}_{n=1}^{\infty}, \left[\sum_{n=1}^{\infty} u_n \text{ converges} \right]$

30. $\forall \{u_n\}_{n=1}^{\infty}, \left[\lim_{n\to\infty} u_n = 0 \to \sum_{n=1}^{\infty} u_n \text{ converges} \right]$

31. $\forall f, \left(f \text{ is continuous} \to f \text{ is differentiable} \right)$

32. $\forall f,\ f$ is bounded

CHAPTER 2 PROOF

2.1 MATHEMATICAL SYSTEMS

A ***mathematical system*** consists of the following:

a) a set of undefined concepts;

b) a universal set;

c) a set of relations (we will define “relation” later);

d) a set of operations (we will define “operation” later);

e) a set of logical axioms (the rules of reasoning—logic);

f) a set of non-logical axioms (these axioms pertain to the elements, relations, and operations; the entities studied by mathematicians; such an axiom might be $a+b=c+b \to a=c$);

g) a set of theorems;

h) a set of definitions;

i) an underlying set theory. (We will study this in Chapter 3.)

For example, in plane geometry the undefined concepts were those of *point* and *line*. The universal set was the set of points in the plane. The relations were such concepts as equality, perpendicularity, and parallelism. We have already studied the logical axioms. An example of a non-logical axiom is:

Two different points are on exactly one line.

Another example of a mathematical system is the system of real numbers, some axioms and theorems of which are considered in the Appendix. In this chapter we will consider proofs in this system. Our emphasis will be on the ways of going about a proof.

Every mathematical discourse is in reference to some mathematical system, even though it may not be clearly specified.

Definitions.

> "When I use a word," Humpty Dumpty said in a rather scornful tone, "it means just what I said it to mean—neither more nor less."
>
> Lewis Carroll, *Through the Looking Glass*

A ***definition*** is an abbreviation. As abbreviations, definitions can be short, for example,

$a < b$ iff $b > a$,

or long, for example,

f is ***integrable*** on $[a, b]$ iff $\lim\limits_{\substack{x_j - x_{j-1} \to 0 \\ n \to \infty}} \sum\limits_{j=1}^{n} f\left(\delta_j\right)\left(x_j - x_{j-1}\right) < \infty.$

You can always substitute an expression being defined for that which defines it, and conversely.

You should learn to read into a definition its "iff," or "equivalence" meaning. Definitions are often stated in a manner which conceals the possibility of substituting one expression for another due to an intended but unstated equivalence. For example, consider the definitions:

An ***even integer*** a is of the form $a = 2k$, k an integer.

An integer a is ***even*** if $a = 2k$, k an integer.

Each could be restated:

An integer a is ***even*** iff $a = 2k$, k an integer.

Then, by the Rule of Substitution, either expression could be substituted for the other.

Exercise Set 2.1

Restate the definitions in 1 through 8 in "iff" form.

1. If $a = 2k + 1$, then a is ***odd***.

2. A ***quadrilateral*** is a polygon with just four sides.

3. The ***maximum value of f on S***, denoted $\max_S f$, is the largest value assumed by f on S.

4. A series is ***divergent*** if it is not convergent.

5. A ***triangle*** is a polygon with just three sides.

6. A ***real number*** is a number x that is equal to an infinite decimal.

7. A real number x which is not a rational number is an ***irrational number***.

8. The ***complex numbers*** are the numbers of the form $x + yi$, where x and y are real numbers and $i^2 = -1$.

9. Find three examples of incorrectly stated definitions in mathematics textbooks.

10. Find a textbook where the student has first been taught the meaning of "iff" and in which most definitions are stated as equivalences.

11. Look up the definition of triangle congruence in a geometry text. Would you call this a long or short definition?

2.2 PROOF

Definition. Suppose $A_1, A_2, \ldots, A_k$ are all the axioms and previously proved theorems of a mathematical system. A ***formal proof*** (or deduction) of a sentence *P* is a sequence of statements $S_1, S_2, \ldots, S_n$, where

(1) S_n is *P* (the last statement is *P*),

and one of the following holds:

(2) S_i is one of $A_1, A_2, \ldots, A_k$;

(3) S_i follows from the previous statements by a valid argument using the rules of reasoning.

A ***theorem*** is any sentence deduced from the axioms and/or the previous theorems.

The definition of formal proof is somewhat complicated. An example should help.

EXAMPLE. Suppose a mathematical system contains just the following axioms:

$$A_1 : a + b = c \to \left[x < y \wedge (2 = 3)\right]$$
$$A_2 : a + b = c$$

The following is a formal proof of $x < y$:

$S_1 : a + b = c \to (x < y \wedge 2 = 3)$, by A_1

$S_2 : a + b = c$, by A_2

$S_3 : x < y \wedge 2 = 3$, by modus ponens on S_1, S_2

$S_4 : x < y$, by the tautology $(P \wedge Q) \to P$

In practice mathematicians do not write formal proofs. They write informal proofs. An ***informal proof*** is an argument which shows the existence of a formal proof. As such it gives enough of the formal proof so that another person becomes "convinced." Thus, we might call an informal proof a "convincing argument." Mathematicians try to convince other mathematicians. You will try to convince your fellow students and your instructor.

EXAMPLE. The following is an informal proof of $x < y$, in the previous system:

Informal proof. From A_1 and A_2 it follows that $x < y \wedge (2 = 3)$. Thus $x < y$.

Henceforth we will be writing only informal proofs. The art of mathematics is creating proofs. Just as a painter has some basic modes of painting, such as oils, water colors, and wood cuts; so the mathematician has some basic modes of proof. We now consider these modes of proof.

Exercise Set 2.2

In the mathematical system of the preceding example, give

1. a formal proof of 2 = 3.
2. an informal proof of 2 = 3.

2.3 PROVING SENTENCES OF THE TYPE $P \to Q$

Now we consider two modes of proof for sentences of the type $P \to Q$; later we consider others.

Rule of Conditional Proof—RCP. You usually proved a sentence of the type $P \to Q$ in plane geometry by assuming P and deducing Q. You considered Q the conclusion. In actuality

$$P \to Q$$

was the conclusion; it was what you were trying to prove.

To prove $P \to Q$, first assume P to be true (make it an axiom temporarily). Then using P and any of the other theorems and axioms try to deduce Q. Once Q is deduced in this manner, you have completed a proof of $P \to Q$. You have not shown that Q is true; you have only shown that Q is true if P is true. Whether P is true is another question; whether Q is true is also another question. What you have shown to be true is $P \to Q$.

To explain this more formally, suppose $A_1, \ldots, A_n$ are the axioms and previously-proved theorems. To prove $P \to Q$ is to show that

$$\text{From } A_1, \ldots, A_n \text{ we can deduce } P \to Q$$

is a valid argument. To do this temporarily assume P is an axiom and show that

$$\text{From } A_1, \ldots, A_n, P \text{ we can deduce } Q$$

is a valid argument. The above is referred to as **The Deduction Theorem**, though we consider it a proof axiom.

EXAMPLE. Recall: a is an even integer iff a can be expressed in the form $a = 2k$, where k is some integer.

Prove: a is an even integer → a^2 is an even integer.

Proof. Assume a is an even integer. Then $a = 2k$ for some integer k. Hence $a^2 = 2(2k^2)$ and $2k^2$ is an integer, so a^2 is even. ■

Within the previous proof we used the tautology

$$[(P \to S_1) \wedge (S_1 \to S_2) \wedge \cdots \wedge (S_n \to R)] \to (P \to R).$$

That is,

$$a \text{ even} \to a = 2k \to a^2 = 2(2k^2) \to a^2 \text{ is even;}$$

$$\therefore a \text{ even} \to a^2 \text{ is even.}$$

All proofs will be given in a paragraph style because this is the way experienced mathematicians write proofs. This style differs from the more-difficult-to-write parallel column format sometimes used in plane geometry.

The Rule of Conditional Proof actually provides another (assumed) way to establish that a conditional sentence is true. To explain this, note that the sentence

If grass is red, then 3 = 4 (1)

is true because the antecedent is false. This might be called "structural truth." Compare this with the sentence

If $4x + 5 = 13$, then $x = 2$. (2)

Now any replacement for x which makes the antecedent $4x + 5 = 13$ false, makes the sentence (2) true. So the only concern is whether (2) is true when $4x + 5 = 13$ is true. To establish that (2) is true, one establishes the truth of $x = 2$ based on the truth of $4x + 5 = 13$. This might be called "truth by **dependence** of the consequent on the antecedent."

As another example of the Rule of Conditional Proof (RCP), we deduce another reasoning sentence.

THEOREM. Every sentence of the type

$$\forall x P(x) \to \exists x P(x)$$

is true.

Proof. Assume $\forall x P(x)$ is true. Then the solution set for $P(x)$ is the universal set. Since universal sets are assumed to be nonempty, the sentence $\exists x P(x)$ is true. ■

Is it worth asking why $\forall x P(x) \to \exists x P(x)$ is true when $\forall x P(x)$ is false? The sentence is true by the truth table for '→'. That is, the antecedent is false so the conditional is true.

Proving $P \to Q$ by Contrapositive.

We can prove

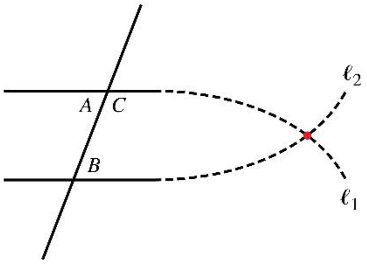
$$P \to Q$$

by proving its contrapositive

$$\sim Q \rightarrow \sim P.$$

The two are equivalent.

EXAMPLE. The following proof is from Euclidean geometry. We assume the student has a knowledge of its axioms and properties.



Prove: $\angle A = \angle B \rightarrow \ell_1 \cap \ell_2 = \varnothing.$

Proof. By contrapositive we will prove

$$\ell_1 \cap \ell_2 \neq \varnothing \rightarrow \angle A \neq \angle B.$$

Assume $\ell_1 \cap \ell_2 \neq \varnothing;$ that is, they intersect at a point R. Then RCB is a triangle, so

$\angle C + \angle B + \angle R = 180^\circ$. Also $\angle A$ and $\angle C$ are supplementary. Hence $\angle C + \angle B + \angle R =$

$\angle A + \angle C$, so $\angle B + \angle R = \angle A$. Recall that the measure of any angle of a triangle is positive so $\angle R > 0$. Then $\angle B < \angle A$ (see P1 of the Appendix) or $\angle A \neq \angle B$. ■

Notice that in the previous proof the rule of conditional proof was used to prove the contrapositive.

In summary, we have considered two ways of proving $P \rightarrow Q$.

1) **RCP**. Assume P, deduce Q.

2) **Contrapositive**. Prove $\sim Q \rightarrow \sim P$; assume $\sim Q$ and deduce $\sim P$.

Later we consider others.

Exercise Set 2.3

Consider

$$a^2 \text{ is an even integer} \rightarrow a \text{ is an even integer}$$

1. State the contrapositive.

2. Prove the contrapositive.

Give a direct proof of each of the following using RCP.

3. If a is even and b is even, then $a + b$ is even.

4. If a is even and b is even, then ab is even.

5. If a is even and b is odd, then $a + b$ is odd.

6. If a is even and b is odd, then ab is even.

7. If a is odd and b is odd, then $a + b$ is even.

8. If a is odd and b is odd, then ab is odd.

9. If a is odd, then a^2 is odd. (Although you are asked to give a proof here, do you see why it is unnecessary based on what we have already proved?)

10. Prove: Every sentence of the type

$$\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$$

is true.

11. Give a proof by contrapositive of the sentences in Exercises 9 and 10.

12. If a^2 is odd, then a is odd. (Again, though you are asked to give the proof, do you see why it is unnecessary?)

13. A **proper divisor** of a number is a divisor which is less than the number. A **perfect** number is a number which is the sum of its proper divisors. For example, 6 is a perfect number.

Prove: If n (natural number) is perfect, then n is not prime.

14. Mathematicians often prove a sentence of the type $P \rightarrow (Q \wedge R)$ by proving $P \rightarrow Q$ and $P \rightarrow R$. Find a tautology which justifies this.

15. Mathematicians often prove a sentence of the type $P \to (Q \to R)$ by proving $(P \wedge Q) \to R$. Find a tautology which justifies this.

16. Mathematicians often prove a sentence of the type $(P \to Q) \to (S \to R)$ by proving

$[(P \to Q) \wedge S] \to R$. Justify this with a tautology.

2.4 PROVING SENTENCES OF THE TYPE $P \leftrightarrow Q$

We consider three modes of proof for sentences of the type $P \leftrightarrow Q$.

Prove $P \to Q$ and $Q \to P$. One mode of proof for

$$P \leftrightarrow Q$$

is derived from its definition:

$$(P \to Q) \wedge (Q \to P).$$

Thus there are two steps in the proof:

a) **Prove $P \to Q$;** referred to as the "only if," or "sufficiency," part.

b) **Prove $Q \to P$;** referred to as the "if," or "necessity," part.

Each of these sentences is a conditional which might be proved using previously considered modes of proof.

EXAMPLE. Prove: Real numbers a and b are roots of the equation $x^2 + px + q = 0$ iff $a + b = -p$ and $ab = q$.

Proof.

a) (Only if, or sufficiency)

Prove: If a and b are roots of the equation $x^2 + px + q = 0$, then $a + b = -p$ and $ab = q$

Using RCP, assume a and b are roots of the equation. Then via the quadratic formula we know that

$$a = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } b = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

(The signs +, − could be interchanged without affecting the proof.)

Then $a + b = -p$ and $ab = q$, by algebra.

b) (If, or necessity)

Prove: If $a + b = -p$ and $ab = q$, then a and b are roots of the equation

$x^2 + px + q = 0$.

Again, using RCP assume $a + b = -p$ and $ab = q$. Then $b = -p - a$ and $q = ab = a(-p - a) = -ap - a^2$. Hence $a^2 + pa + q = 0$, so a is a root of $x^2 + px + q = 0$. Similarly, interchanging a and b in the argument shows that a is also a root of $x^2 + px + q = 0$. ■

Prove $P \to Q$ and $\sim P \to \sim Q$.

Another mode of proof for

$P \leftrightarrow Q$

is to prove

$P \to Q$,

as before, but then prove the contrapositive of $Q \to P$,

$\sim P \to \sim Q$.

For example, suppose you wanted to prove

a is even iff a^2 is even,

using the mode of proof just described. The sentences to be proved are:

a) a even → a^2 even;

b) a not even (odd) → a^2 not even (odd).

We have proved these sentences in previous work.

Iff-String. A third mode of proof for

$P \leftrightarrow Q$

is accomplished by producing a string of equivalent sentences leading from P to Q as follows.

$P \leftrightarrow Q_1$		$P \leftrightarrow Q_1$
$Q_1 \leftrightarrow Q_2$	abbreviated	$\leftrightarrow Q_2$
$\vdots$		$\vdots$
$Q_n \leftrightarrow Q$		$\leftrightarrow Q$

Once each of the previous is proved $P \leftrightarrow Q$ follows by the tautology

$$\left[(P \leftrightarrow Q_1) \wedge \cdots \wedge (Q_n \leftrightarrow Q)\right] \to (P \leftrightarrow Q).$$

EXAMPLE. Prove. Every sentence of the type

$$\forall x \forall y P(x, y) \leftrightarrow \forall y \forall x P(x, y)$$

is true.

Proof. Using Iff-string we have

$\forall x \forall y P(x, y)$ is true $\leftrightarrow$ for every replacement of x and y by members a and b of the universal set $P(a, b)$ is true;

$\leftrightarrow$ for every replacement of y and x by members a and b of the universal set $P(a, b)$ is true;

$\leftrightarrow$ $\forall y \forall x P(x, y)$ is true. ■

There is another aspect of "Iff-string." To prove $P \leftrightarrow Q$ by means of a string it suffices to prove $P \to Q_1, Q_1 \to Q_2, \ldots, Q_n \to Q,$ for some $Q_1, \ldots, Q_n,$ and $Q \to S_1, S_1 \to S_2, \ldots, S_k \to P,$ for some $S_1, \ldots, S_k$. In a similar manner, one sometimes has to prove

P, Q, S, T

equivalent. A way this might be proved is to prove

$P \to Q \to S \to T \to P,$

and thus cut almost in half the number of proofs otherwise encountered in proving $P \leftrightarrow Q,$ $Q \leftrightarrow S, S \leftrightarrow T.$

Many more examples of this mode of proof will be given in Chapter 3 on set theory.

In summary, we have considered three modes of proof for sentences of the type $P \leftrightarrow Q$:

a) **Prove $P \to Q$ and $Q \to P$.** ($Q \to P$ is called the **converse** of $P \to Q$.)

b) **Prove $P \to Q$ and $\sim P \to \sim Q$.** ($\sim P \to \sim Q$ is called the **inverse** of $P \to Q$.)

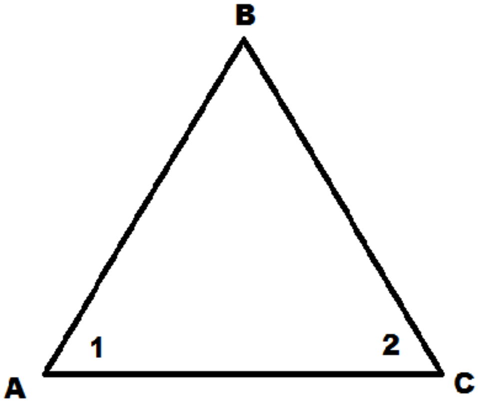
c) **Iff-String.** Produce a string of equivalent sentences leading from P to Q.

Exercise Set 2.4

Prove the sentences in Exercises 1 through 7.

1. a is an odd integer $\leftrightarrow$ a^2 is an odd integer.

2. Consider the figure



Prove: $AB = BC \leftrightarrow \angle 1 = \angle 2$

Use congruence properties of geometry.

3. $a < b \leftrightarrow a + c < b + c$. Use the real number properties of the Appendix.

4. x is an odd integer iff $x + 1$ is an even integer.

5. x is an even integer iff $x + 2$ is an even integer.

6. Every sentence of the type

$$\exists x \exists y P(x, y) \leftrightarrow \exists y \exists x P(x, y)$$

is true.

7. Every sentence of the type

$$\forall x[P(x) \wedge Q(x)] \leftrightarrow [\forall x P(x) \wedge \forall x Q(x)]$$

is true.

8. Mathematicians often prove a sentence of the type $P \rightarrow (Q \leftrightarrow R)$ by proving $(P \wedge Q) \rightarrow R$ and $(P \wedge R) \rightarrow Q$. Find a tautology which justifies this.

2.5 PROVING SENTENCES OF THE TYPE $\forall x P(x)$

To prove

$$\forall x P(x)$$

let x represent an arbitrary element of the universal set and prove

$$P(x)$$

true. Then since x was an arbitrary element of the universal set generalize

$$\forall x P(x)$$

true. This is justified by LOGICAL AXIOM 2, p. 45.

EXAMPLE. Consider

$$\forall f\,(f \text{ is differentiable} \rightarrow f \text{ is continuous}).$$

To prove the sentence let f be an arbitrary function and prove

$$f \text{ is differentiable} \rightarrow f \text{ is continuous}.$$

By RCP, assume f differentiable and prove f continuous. We will not include the proof; it appears in most calculus texts. Once we have proved

$$f \text{ is differentiable} \rightarrow f \text{ is continuous},$$

we have proved

$$\forall f\,(f \text{ is differentiable} \rightarrow f \text{ is continuous}),$$

since f was an arbitrary function.

EXAMPLES.

a) Consider $\forall x\left(1 < x \rightarrow 1 < x^2\right)$ with universal set $\{2, 3\}$. Considering

$$1 < 2 \rightarrow 1 < 2^2$$

and

$$1 < 3 \rightarrow 1 < 3^2,$$

we have proved the sentence by substitution.

b) Consider $\forall x\left(1<x\rightarrow 1<x^2\right)$ with infinite universal set ℕ. Trying to prove the sentence by substituting each element would be impossible. To do the proof, let x be arbitrary and prove

$$1<x\rightarrow 1<x^2.$$

Such a proof would depend on the axioms for ℕ (see Appendix) which are stated in terms of quantifiers.

Proof. Assume $1 < x$. Then since $1 > 0$, by O6, it follows that $x > 0$ by O2. Then $1 < x$ and $x > 0$ implies $1\cdot x<x\cdot x,$ by O4. So $1 < x$ and $x < x^2$ implies $1 < x^2$, by O2. Therefore,

$$1<x\rightarrow 1<x^2$$

so

$$\forall x\left(1<x\rightarrow 1<x^2\right)$$

is proved. ■

To prove $\exists x P(x)$ show or prove there exists an x in the universal set for which $P(x)$ is true.

EXAMPLE. Prove: ∃ f (f is continuous ∧ f is **not** differentiable).

Proof. The function described by $f(x) = | x |$ is continuous but not differentiable at $x = 0$. ■

We will comment about another mode of proof for $\exists x P(x)$ when we consider proof by contradiction.

Exercise Set 2.5

Describe the modes of proof you might use to prove each of the following.

1. $\forall x\left(x^2 \text{ is even iff } x \text{ is even}\right)$.
2. $\forall a \forall b\ (a < b \text{ iff } a + 8 < b + 8)$.
3. For any two sets A and B, $x \in A \cup B$ iff $x \in B \cup A$.
4. For any set A, $A \subseteq A$.
5. For any set A, $\emptyset \subseteq A$.
6. $\exists x,\ x^2 = x$

7. $\exists\{u_n\}, \left(\sum_{n=1}^{\infty} u_n \text{ is divergent } \wedge \lim_{n\to\infty} u_n = 0\right)$

8. $\exists y \forall x, x + y = x$

9. $\exists y \forall x, xy = x$

2.6 PROOF BY CASES

Proof by cases is used several ways and involves the connective ∨ (or).

Proving a Sentence of the Type $(P \vee Q) \to R$. This type of proof utilizes the tautology

$$\left[(P \to Q) \wedge (R \to Q)\right] \to \left[(P \vee R) \to Q\right] \tag{1}$$

The proof is accomplished by proving the antecedent of (1),

$$(P \to Q) \wedge (R \to Q).$$

Hence $P \to Q$ and $R \to Q$ must be proved. Any mode of proof for conditional sentences can be used. Intuitively,[*] you want to prove that Q can be deduced from either P or R, so you must show that from either one you can deduce Q.

EXAMPLE. Prove: $(a = 0 \vee b = 0) \to ab = 0$.

Proof.

CASE 1) Prove $a = 0 \to ab = 0$. Assume $a = 0$. Then $ab = 0 \cdot b = 0$, by P3.

CASE 2) Prove $b = 0 \to ab = 0$. The proof is analogous to Case 1. ■

Similarly, a proof by cases of

$$(P_1 \vee \cdots \vee P_n) \to Q$$

is accomplished by proving

$$\begin{aligned} &P_1 \to Q \\ &P_2 \to Q \\ &\quad\vdots \\ &P_n \to Q. \end{aligned}$$

Such a proof has n cases and is justified by the tautology

$$\left[(P_1 \to Q) \wedge \cdots \wedge (P_n \to Q)\right] \to \left[(P_1 \vee \cdots \vee P_n) \to Q\right].$$

As an Intermediary Step. Suppose we are again proving

$$P \to Q.$$

---

[*] Which means "appealing to your mathematical experience."

We might discover that

$$P \to (P_1 \vee P_2 \vee \cdots \vee P_n) \tag{2}$$

and

$$(P_1 \to Q) \wedge (P_2 \to Q) \wedge \cdots \wedge (P_n \to Q).$$

By proof by cases we have shown that

$$(P_1 \vee P_2 \vee \cdots \vee P_n) \to Q. \tag{3}$$

Then using (2) and (3) and The Law of Syllogism it follows that

$$P \to Q.$$

Hence another way to use proof by cases is as an intermediary step derived from the antecedent of a conditional sentence.

EXAMPLE. Recall the definition:

$|x| = x$ when $x \geq 0$,
$|x| = -x$ when $x < 0$.

Prove: If x is a real number, then $|x| \geq 0$.

Proof. If x is a real number, then $x \geq 0 \vee x < 0$. We will prove

$$(x \geq 0 \vee x < 0) \to |x| \geq 0.$$

CASE 1) $x \geq 0$. If $x \geq 0$, then by definition $|x| = x$ so $|x| \geq 0$.

CASE 2) $x < 0$. If $x < 0$, then by definition $|x| = -x$. By properties of inequalities if $x < 0$, then $-x > 0$ so $|x| > 0$.

Hence $(x \geq 0 \vee x < 0) \to |x| \geq 0$. Therefore, if x is a real number, then $|x| \geq 0$. ■

The art of producing a proof by cases may be discovering what set of exhaustive cases is appropriate. For example, if x is a real number you might use

a) $x \geq 0$ or $x < 0$;

b) $x > 0 \vee x = 0 \vee x < 0$;

c) $x > 2 \vee x = 2 \vee x < 2$.

Notice each example is exhaustive in that all possibilities occurred. As other examples, a function is either continuous or discontinuous, an integer is either odd or even.

Exercise Set 2.6

Complete.

1. If A is an angle, the cases you might consider are A acute $\vee$ ____________ $\vee$_____________.
2. If f is a function, the cases you might consider are:
   a) f is differentiable $\vee$ ______________.
   b) f is even $\vee$ _____________ $\vee$ f is neither even nor odd.
   c) f is constant $\vee$ _____________.
3. If x is an integer the cases you might consider are:
   a) x is even $\vee$ _____________.
   b) $x > 9 \vee$ _____________ $\vee\ x < 9$.

Use proof by cases to prove the following.

4. If x is a real number, then $|-x| = |x|$.
5. If x is a real number, then $|x^2| = |x|^2$.
6. For every real number x, $x \le |x|$.
7. If x and y are real numbers, then $|xy| = |x| \cdot |y|$. *Hint:* $x \ge 0 \wedge y \ge 0$, $x < 0 \wedge y < 0$, $x \ge 0 \wedge y < 0$, $x < 0 \wedge y \ge 0$.
8. If $a > 0$, then $|x| < a$ iff $-a < x < a$.
9. If $a > 0$, then $|x| > a$ iff $x > a \vee x < -a$.
10. If x and y are real numbers, then $|x+y| \le |x| + |y|$.
11. If x and y are real numbers, then $|x| - |y| \le |x-y|$.
12. If f is a strictly monotone function, then f is one-to-one. *Hint:* f is strictly monotone $\rightarrow f$ is strictly increasing or strictly decreasing.
13. If x is an integer, then $x^2 - x$ is even.

14. If x is an integer, then $x^2 + x + 1$ is odd.

15. Find a proof by cases of the Law of Cosines in a trigonometry book. State the proof and explain the use of Proof by Cases.

16. Find a proof by cases of the formula

$$\sin\left(a+b\right) = \sin a \cos b + \cos a \sin b$$

in a trigonometry book. State the proof and explain the use of Proof by Cases.

17. Find a proof by cases of Rolle's Theorem in a calculus book. State the proof and explain the use of Proof by Cases.

18. The function g described by $g(x) = | x |, x \neq 0$, is differentiable while the function f described by $f(x) = | x |$ is not differentiable. Find a formula for g' using proof by cases. Use a calculus book if necessary.

19. <u>Prove</u>: Every sentence of the type

$$\left[\forall x P\left(x\right) \vee \forall x Q\left(x\right)\right] \rightarrow \forall x\left[P(x) \vee Q(x)\right]$$

is true.

20. Suppose you want to prove a sentence of the type

$$P \rightarrow (R \wedge Q)$$

by contrapositive. Explain the possible role of proof by cases in such a proof.

2.7 MATHEMATICAL INDUCTION

Consider proving sentences of the type

For every natural number n, $P(n)$

or

$\forall nP(n)$

where the quantifier refers to the set $\mathbb{N} = \{1, 2, 3, \dots\}$. One way to prove sentences of this type is by **mathematical induction,** which uses a rule of reasoning not yet discussed. The following is the mathematical induction sentence, which the mathematician *accepts* as an axiom. (This is another situation in which the mathematician makes assumptions about how he will reason.)

Principle of Mathematical Induction. Suppose $P(n)$ is a sentence which is a statement for any $n \in \mathbb{N}$, then

$$\left[P(1) \wedge \forall k, P(k) \to P(k+1)\right] \to \forall nP(n) \qquad \text{(MI)}$$

If we can prove the antecedent of MI,

$$P(1) \wedge \forall k\left[P(k) \to P(k+1)\right],$$

then by modus ponens we can deduce

$\forall nP(n)$.

Thus there are two steps in the proof of $\forall nP(n)$:

1) BASIS STEP: Prove $P(1)$.

2) INDUCTION STEP: Prove $\forall k, P(k) \to P(k+1)$.

That is, we prove $P(1)$ and for every k, $P(k) \to P(k+1)$.

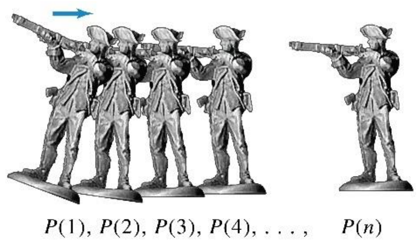
To explain intuitively how this proves $\forall nP(n)$ suppose we have completed both parts of the induction proof; that is $P(1)$ and $\forall k, P(k) \to P(k+1)$ are proved. We have deduced an endless sequence of sentences

$$\left.\begin{array}{l} P(1) \\ P(1)\to P(2) \\ P(2)\to P(3) \\ \qquad\vdots \\ P(n-1)\to P(n) \\ \qquad\vdots \end{array}\right\} \qquad \forall k, P(k)\to P(k+1)$$

The process then becomes similar to knocking over a row of tin soldiers for

$$\begin{array}{l} P(1) \\ \underline{P(1)\to P(2)} \\ \therefore P(2) \end{array} \text{ , then } \begin{array}{l} P(2) \\ \underline{P(2)\to P(3)} \\ \therefore P(3) \end{array} \text{ , then } \begin{array}{l} P(3) \\ \underline{P(3)\to P(4)} \\ \therefore P(4) \end{array} \text{ , and so on, producing the}$$

endless sequence $P(1), P(2), \ldots, P(n), \ldots;$



that is, we have proved $\forall n P(n)$. We will be using basic algebra in the following proofs.

EXAMPLE. Prove: $\forall n,\ 2^n < 2^{n+1}$

Proof. $P(n)$: $2^n < 2^{n+1}$

1) BASIS STEP: Prove $P(1)$: $2^1 < 2^{1+1}$

Now $2^1 = 2, 2^{1+1} = 4,$ so $2^1 < 2^{1+1}$.

2) INDUCTION STEP. Prove $\forall k,\ P(k) \to P(k+1)$

Assume $P(k)$: $2^k < 2^{k+1}$

Deduce $P(k + 1)$: $2^{k+1} < 2^{k+2}$

Then $2^k < 2^{k+1}$, by $P(k)$

$2 \cdot 2^k < 2 \cdot 2^{k+1}$, by Property O4 of the Appendix

$2^{k+1} < 2^{k+2}$

Thus *P*(*k* + 1) follows. ■

Note how the proof was created; we saw that multiplying both sides of the inequality *P*(*k*) by 2 gave the inequality *P*(*k* + 1).

When doing a proof by mathematical induction it is helpful to list

P(*n*),

P(1),

P(*k*),

P(*k* + 1)

as illustrated in the previous example. This aids in identifying what is to be assumed and what is to be proved. Usually proving *P*(1) is just a matter of substitution, but proving ∀*k*, *P*(*k*) →*P*(*k* +1) requires more effort. An aid to doing this is having listed *P*(*k*) and *P*(*k* + 1), examine *P*(*k* + 1) and try to discover some way of deriving it from *P*(*k*).

We can also use mathematical induction to make definitions, called ***recursive definitions***. For example, the following is a definition of **Σ-** (sigma) **notation**.

Definition. For any natural number *n* and any numbers $a_1, \ldots, a_n$, $\sum_{j=1}^{n} a_j$ is defined as follows:

$$\text{(1)} \quad \sum_{j=1}^{1} a_j = a_1$$

$$\text{(2)} \quad \sum_{j=1}^{k+1} a_j = \left(\sum_{j=1}^{k} a_j \right) + a_{k+1}$$

This definition avoids the somewhat mysterious use of dots, such as

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

and is thus more elegant.

EXAMPLES.

a) $\sum_{j=1}^{4} 2j = \left(\sum_{j=1}^{3} 2j\right) + 8 = \left(\sum_{j=1}^{2} 2j\right) + 6 + 8 = \left(\sum_{j=1}^{1} 2j\right) + 4 + 6 + 8 = 2 + 4 + 6 + 8$

b) $\sum_{j=1}^{5} 3^j = 3^1 + 3^2 + 3^3 + 3^4 + 3^5$

Now let us do another mathematical induction proof.

EXAMPLE. Prove: For every natural number *n*, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. $P(n)$: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

1) BASIS STEP. Prove $P(1)$: $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2+1)}{6}$

This follows by substitution: $1^2 = \frac{1(1+1)(2+1)}{6}$

2) INDUCTION STEP.

Assume $P(k)$: $\sum_{j=1}^{k} j^2 = \frac{k \cdot (k+1) \cdot (2k+1)}{6}$

Deduce $P(k + 1)$:

$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \frac{(k+1)\cdot(k+2)\cdot\left[2(k+1)+1\right]}{6} \\ &= \frac{(k+1)\cdot(k+2)\cdot(2k+3)}{6}\end{aligned}$$

Now

$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2, \text{ by definition of } \Sigma\text{-notation.} \\ &= \frac{k\cdot(k+1)\cdot(2k+1)}{6} + (k+1)^2, \text{ by } P(k)\end{aligned}$$

$$
\begin{aligned}
&= (k+1)\cdot\left[\frac{k\cdot(2k+1)}{6}+(k+1)\right] \\
&= (k+1)\cdot\left[\frac{k\cdot(2k+1)}{6}+\frac{6(k+1)}{6}\right] \\
&= (k+1)\cdot\left[\frac{2k^2+7k+6}{6}\right] \\
&= \frac{(k+1)\cdot(k+2)\cdot(2k+3)}{6}.
\end{aligned}
$$

■

The following is another recursive definition from calculus.

Definition. Suppose *y* is a real function. $D^n(y)$ represents the *n*th derivative of *y* with respect to *x* and is defined as follows:

1) $D^1(y) = D(y)$,

2) $D^{k+1}(y) = D\left[D^k(y)\right]$, the (*k* + 1)st derivative is the derivative of the *k*th derivative.

Now let us do a mathematical induction proof from calculus.

EXAMPLE. Prove: For every natural number *n*, $D^n\left(xe^x\right) = (x+n)e^x$.

Proof. $P(n)$: $D^n\left(xe^x\right) = (x+n)e^x$

1) BASIS STEP. Prove *P*(1): $D\left(xe^x\right) = (x+1)e^x$

Using the product rule for derivatives we have

$D\left(xe^x\right) = xe^x + e^x = (x+1)e^x$,

so *P*(1) is true.

2) INDUCTION STEP. Prove ∀*k*, *P*(*k*) → *P*(*k* + 1).

Assume *P*(*k*): $D^k\left(xe^x\right) = (x+k)e^x$

Deduce *P*(*k* + 1): $D^{k+1}\left(xe^x\right) = \left[x+(k+1)\right]e^x$.

Now $D^{k+1}\left(xe^x\right)=D\left[D^k\left(xe^x\right)\right]$, since from calculus we know $D^{k+1}=D\left(D^k\right)$

$$\begin{aligned}&=D\left[\left(x+k\right)e^x\right], \text{ by } P\left(k\right)\\&=\left(x+k\right)e^x+e^x, \text{ by the product rule for derivatives}\\&=\left[x+\left(k+1\right)\right]e^x\end{aligned}$$

Hence $P(k + 1)$. ■

Note how important the recursive definitions were in each of the two previous proofs.

It is important to realize that mathematical induction can be applied to prove any sentence $\forall nP(n)$ which refers to the natural numbers. Whether the proof can be accomplished is another problem.

Both the basis and induction steps are essential in a proof by mathematical induction. There are sentences $P(n)$ for which

$P(1)$ is true

but

$\forall nP(n)$ and $\forall k$, $P(k) \rightarrow P(k + 1)$ are *false*.

Such a sentence is $n^2 = n$.

There are sentences $P(n)$ for which

$\forall k\left[P\left(k\right)\rightarrow P\left(k+1\right)\right]$ is true

but

$P(1)$ and $\forall nP(n)$ are *false*.

Such a sentence is $n = n + 1$.

In such cases we can line up the tin soldiers but we cannot knock over the first one.

Mathematical induction can also be used to prove sentences referring to certain subsets of the integers.

Mathematical Induction. For any universal set of the type

$$\{x \mid x \in \mathbb{Z} \text{ and } m \leq x, \text{ for some } m \in \mathbb{Z}\}$$

and any sentence $P(x)$,

$$\left[P(m) \wedge \forall k \geq m, P(k) \to P(k+1)\right] \to \forall x P(x).$$

For example, for the set $\{-3, -2, -1, 0, 1, \ldots\}$ we would have to prove $P(-3)$ and $\forall k \geq -3, P(k) \to P(k+1)$. For $\{0, 1, 2, 3, \ldots\}$ we would have to prove $P(0)$ and $\forall k \geq 0, P(k) \to P(k+1)$. This is illustrated in the next example. First we need another recursive definition.

Definition. For any natural number n and any numbers $a_1, \ldots, a_n$, $\prod_{j=1}^{n} a_j$ is defined as follows:

$$1) \quad \prod_{j=1}^{1} a_j = a_1$$

$$2) \quad \prod_{j=1}^{k+1} a_j = \left(\prod_{j=1}^{k} a_j\right) \cdot a_{k+1}$$

EXAMPLES.

$$\begin{aligned} \text{a)} \quad \prod_{j=1}^{4} \cos j\pi &= \left(\prod_{j=1}^{3} \cos j\pi\right) \cdot \cos 4\pi \\ &= \left(\prod_{j=1}^{2} \cos j\pi\right) \cdot \cos 3\pi \cdot \cos 4\pi \\ &= \left(\prod_{j=1}^{1} \cos j\pi\right) \cdot \cos 2\pi \cdot \cos 3\pi \cdot \cos 4\pi \\ &= \cos \pi \cdot \cos 2\pi \cdot \cos 3\pi \cdot \cos 4\pi \end{aligned}$$

$$\text{b)} \quad \prod_{j=2}^{5} \left(1 - \frac{1}{j}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right)$$

Look for a pattern in the following and try to conjecture a formula for $\prod_{j=2}^{n}(1-1/j)$.

a) $\prod_{j=2}^{2}\left(1-\frac{1}{j}\right)=\left(1-\frac{1}{2}\right)=\frac{1}{2}$

b) $\prod_{j=2}^{3}\left(1-\frac{1}{j}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\frac{1}{3}$

c) $\prod_{j=2}^{4}\left(1-\frac{1}{j}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)=\frac{1}{4}$

In the preceding you made some observations, and based on the observations you hopefully made a conjecture about a formula for $\prod_{j=2}^{n}(1-1/j)$. This kind of reasoning is called ***inductive reasoning***. It uses your intuition (your mathematical experience). You have not yet used **deductive reasoning** to prove your conjecture. Note the following:

1 + 5 = 6

19 + 13 = 32

5 + 7 = 12

Use inductive reasoning and make a conjecture. It is that “the sum of two odd numbers is even.” Earlier we proved this using deductive reasoning. Below we will use “mathematical induction” to give a deductive proof that

$$\prod_{j=2}^{n}\left(1-\frac{1}{j}\right)=\frac{1}{n}.$$

The point here is that “mathematical induction” is somewhat misnamed because it is very much **deductive reasoning**.

EXAMPLE. <u>Prove</u>: For any natural number $n \geq 2$, $\prod_{j=2}^{n}(1-1/j)=1/n$

Proof. The sentence $P(n)$ is $\prod_{j=2}^{n}(1-1/j)=1/n$, with universal set $\{2,3,4,\ldots\}$. Thus we must prove $P(2)$ and for every $k \geq 2$, $P(k)\to P(k+1)$.

1) BASIS STEP. Prove $P(2)$: $\prod_{j=2}^{2}(1-1/j)=1/2$. This follows since 1 – ½ = ½.

2) INDUCTION STEP. Assume $P(k)$: $\prod_{j=2}^{k}(1-1/j)=1/k$. Prove $P(k+1)$: $\prod_{j=2}^{k+1}(1-1/j)=1/(k+1)$.

$$
\begin{aligned}
\text{Now } \prod_{j=2}^{k+1}\left(1-\tfrac{1}{j}\right) &= \prod_{j=2}^{k}\left(1-\tfrac{1}{j}\right)\cdot\left(1-\tfrac{1}{k+1}\right) \text{ by the recursive definition} \\
&= \tfrac{1}{k}\cdot\left(1-\tfrac{1}{k+1}\right) \text{ by } P(k) \\
&= \tfrac{1}{k}\cdot\left(\tfrac{k+1}{k+1}-\tfrac{1}{k+1}\right) \\
&= \tfrac{1}{k+1}.
\end{aligned}
$$

■

Exercise Set 2.7

Prove the following by mathematical induction.

1. $\forall n \in \mathbb{N}, 2^n > n$
2. $\forall n \in \mathbb{N}, 3^n > n$
3. $\forall n \in \mathbb{N}, 2 \le 2^n$
4. $\forall n \in \mathbb{N}, 2n \le 2^n$ (*Hint:* Use Exercise 3)
5. $\forall n \in \mathbb{N}, n < n + 1$
6. $\forall n \in \mathbb{N}, 2^{n-1} \le n!$ (*Hint:* $1! = 1$, $(k + 1)! = (k + 1)k!$)
7. $\forall n \ge 4, 2^n < n!$ Prove $2^n < n!$ first when $n = 3$.
8. Let a and b be positive real numbers. Prove: $\forall n \in \mathbb{N}\ (a < b \to a^n < b^n)$.
9. $\forall n \in \mathbb{N}, (2n)! < 2^{2n}(n!)^2$
10. $\forall n \in \mathbb{N}, \left|\sin(nx)\right| \le n\left|\sin x\right|$ (*Hint:* Use $\sin(a+b)=\sin a\cos b+\cos a\sin b$)
11. **DeMoivre's Theorem:** $\forall n \in \mathbb{N}, \forall u \in \mathbb{R}$, $(\cos u + i\sin u)^n = \cos(nu) + i\sin(nu)$, where $i^2 = -1$
12. $\forall n \in \mathbb{N}, \prod_{j=2}^{n}\left(1-\dfrac{1}{j^2}\right)=\dfrac{n+1}{2n}$

13. $\forall n \in \mathbb{N}, \prod_{j=2}^{n} \cos 2^{j-1} u = \frac{\sin 2^n u}{2^n \sin u}, \sin u \neq 0$

14. **Bernoulli's Inequality:** $a > -1 \to \forall n \in \mathbb{N}, (1+a)^n \geq 1 + an$

15. $\forall n \in \mathbb{N}, D(x^n) = nx^{n-1}$ (*Hint:* Assume $x^0 = 1$ and use the product rule for derivatives)

16. $\forall n \in \mathbb{N}, D^n(\log_e x) = (-1)^{n+1}(n-1)!x^{-n}$

17. $\forall n \geq 5, 2^n > n^2$

18. $\forall n \geq 6, n^3 < n!$

19. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} j = \frac{n^2+n}{2}$; that is, $1 + 2 + \cdots + n = \frac{n^2+n}{2}$

20. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} 2^j = 2^{n+1} - 2$

21. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$

22. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} j(j+1) = \frac{n(n+1)(n+2)}{3}$

23. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} \frac{j}{(j+1)!} = 1 - \frac{1}{(n+1)!}$

24. $\forall n \in \mathbb{N}, \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$

25. $\forall n \in \mathbb{N}, \sum_{j=2}^{n} \frac{1}{\sqrt{j}} > \sqrt{n}$

26. Given the real numbers $a_1, \ldots, a_n$ and using $|x+y| \leq |x| + |y|$, prove $\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$.

27. Given a set of n points in the plane, $n \geq 2$, no three of which are collinear. Prove that the number of straight lines joining these points is $n(n-1)/2$.

28. $\forall m \geq 2$ and $\forall n \in \mathbb{N}$, $m^n > n$. (*Hint:* Let $m \geq 2$ be arbitrary. Then prove $\forall n \in \mathbb{N}$, $m^n > n$ by MI.)

29. $\forall n \in \mathbb{N}$, $1 + r + r^2 + \cdots + r^{n-1} = \dfrac{r^n - 1}{r - 1} \quad (r \neq 1)$. This is the formula for the sum of the terms of a geometric progression.

30. $\forall n \in \mathbb{N}$, $a + (a+d) + \cdots + (a+nd) = \frac{1}{2}(n+1)(2a+nd)$. This is the formula for the sum of the terms of an arithmetic progression.

31. Why cannot the following be proved by MI?

a) $\forall n \in \mathbb{N}$, $3 + 5 + \cdots + (2n+1) = (n+1)^2$

b) $\forall n \in \mathbb{N}$, $1 + 3 + \cdots + (2n-1) = n^2 + 3$

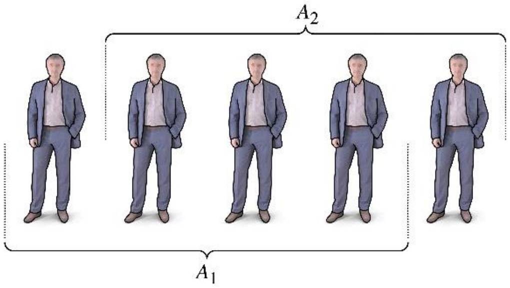
32. Find the error in this proof.

Theorem. Everyone is of the same sex.

Proof. Let $P(n)$ be the following sentence:

If A is a set containing n people, then all the people have the same sex.

Indeed, $P(1)$ is true. Assume $P(k)$ is true. Let A be a set of $k + 1$ people. Then A is the union of two overlapping sets A_1 and A_2 each containing k people. (Consider the following illustration when $n = 5$)



By $P(k)$ all the people in A_1 are of the same sex, and all the people in A_2 are of the same sex. Since A_1 and A_2 overlap all the people in A are of the same sex.

33. Use proof by cases and mathematical induction to prove the following. For every natural number n, $i^n = 1, -1, i$, or $-i$, where $i^2 = -1$.

34. Consider $P(n)$: $n^2 = n$

a) Prove $\forall n P(n)$ is false by finding a counterexample.

b) Prove $P(1)$.

c) Form the negation of $\forall k\left[P(k) \rightarrow P(k+1)\right]$. Prove the negation.

35. Consider $P(n)$: $n = n + 1$

a) Prove $\forall n P(n)$ is false by finding a counterexample.

b) Prove $P(1)$ is false.

c) Prove $\forall k\left[P(k) \rightarrow P(k+1)\right]$.

Prove the following by mathematical induction.

36. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

37. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right)=n+1$

38. $\cos n\pi=(-1)^n$ (*Hint:* Use an identity for $\cos(\alpha+\beta)$.)

39. $\sum_{j=1}^{n}\frac{1}{j(j+1)}=\frac{1}{n+1}$

40. For every natural number $n \geq 2$, $\log_a\left(b_1 b_2\cdots b_n\right)=\log_a b_1+\log_a b_2+\cdots+\log_a b_n$.

Prove the following for any complex numbers $z_1, \ldots, z_n$, where $i^2=-1$ and $\overline{z}$ is the conjugate of z. (If z is the complex number $z = a + bi$, then $\overline{z}=a-bi$.)

41. $\overline{z^n}=\overline{z}^n$

42. $\overline{z_1+z_2+\cdots+z_n}=\overline{z}_1+\overline{z}_2+\cdots+\overline{z}_n$

43. $\overline{z_1\bullet z_2\bullet\cdots\bullet z_n}=\overline{z}_1\bullet\overline{z}_2\bullet\cdots\bullet\overline{z}_n$

44. i^n is either $1, -1, i,$ or $-i$.

For any integers *a* and *b*, *b* is a factor of *a* if there exists an integer *c* such that *a* = *bc*. Prove the following for any natural number *n*.

45. 3 is a factor of $n^3 + 2n$

46. 2 is a factor of $n^2 + n$

47. 5 is a factor of $n^5 + n$

48. 3 is a factor of $n(n+1)(n+2)$

49. **The Tower of Hanoi problem.** There are three pegs on a board. On one peg are *n* disks, each smaller than the one on which it rests. The problem is to move this pile of disks to another peg. The final order must be the same, but you can move only one disk at a time and you can never place a larger disk on a smaller one.

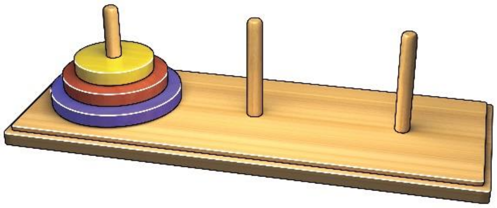
a) What is the least number of moves it takes to move three disks?

b) What is the least number of moves it takes to move four disks?

c) What is the least number of moves it takes to move two disks?

d) What is the least number of moves it takes to move one disk?

e) Conjecture a formula for the least number of moves it takes to move *n* disks. Prove it by mathematical induction.



2.8 PROOF BY CONTRADICTION

“Eliminate all other factors, and the one which remains must be the truth.” (Sherlock Holmes)

Sir Arthur Conan Doyle, The Sign of Four

A ***contradiction*** is a statement which is false no matter what the truth value of its constituent parts. For example, the sentence

$R \wedge \sim R$

is always false as shown in its truth table.

R	$\sim R$	$R \wedge \sim R$
T	F	F
F	T	F

A ***proof by contradiction*** of a sentence

P

is a proof that assumes

$\sim P$

and yields a sentence of the type

$R \wedge \sim R,$

where R is any sentence including P, an axiom, or any previously proved theorem. This is justified by the tautology

$$\left[\sim P \rightarrow \left(R \wedge \sim R\right)\right] \rightarrow P.$$

Intuitively, P can only be true or false, but not both. If we assume its negation true and this yields another sentence both true and false, then $\sim P$ cannot be true so P must be true.

The phrases ***reduction ad absurdum***, meaning “reduce to an absurdity,” and ***indirect proof*** also refer to proof by contradiction.

The importance of being able to form sentence negations is realized when doing proofs by contradiction. To begin such proofs, you must know how to form negations.

EXAMPLES.

To begin a proof by contradiction of

a) $\forall x P(x)$ b) $\exists x P(x)$ c) $P \to Q$ d) $Q \to \exists x P(x)$

we would assume the following negations

a) $\exists x \sim P(x)$ b) $\forall x \sim P(x)$ c) $P \wedge \sim Q$ d) $Q \wedge \forall x \sim P(x)$

Proof by contradiction provides another mode of proof for proving sentences of the type

$\forall x P(x)$

or

$\exists x P(x)$.

In fact, it provides another mode of proof for proving any sentence.

We proved several sentences of the type $\exists x P(x)$ directly, by displaying an x such that $P(x)$ is true. A proof by contradiction of $\exists x P(x)$ may not display an x; that is, one could prove there exists an x without displaying it. This is illustrated in calculus where an indirect proof[*] of the sentence

$$\exists x, \lim_{n\to\infty} \int_1^n e^{-y^2} = x$$

can be given without actually displaying x.

Proofs of existence theorems in differential equations provide other such illustrations.

Proving a Sentence of the Type $P \to Q$ by Contradiction. Most proofs by contradiction are of sentences of the type $P \to Q$. To prove a sentence of the type

$P \to Q$

by contradiction, assume its negation

$\sim (P \to Q)$ or $P \wedge \sim Q$.

Hence assume both P and $\sim Q$ true, and deduce a sentence of the form $R \wedge \sim R$.

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[*] See Thomas, George B., Calculus and Analytic Geometry, Reading, MA, Addison-Wesley Pub. Co., 5th Ed., 1979, p. 379.

EXAMPLE 1. Prove: For every x, $x \neq 0 \rightarrow x^{-1} \neq 0$.

Proof. For contradiction assume the negation

There exists an x such that $x \neq 0 \wedge x^{-1} = 0$.

Then by M4,

$$x \cdot x^{-1} = 1.$$

Also, using $x^{-1} = 0$ and P3,

$$x \cdot x^{-1} = x \cdot 0 = 0.$$

Hence 1 = 0. Thus, the contradiction is

$$1 \neq 0 \wedge 1 = 0.$$

So, for every x,

$$x \neq 0 \rightarrow x^{-1} \neq 0. \ \blacksquare$$

EXAMPLE 2. Prove: For every x and every y, if x is rational and y is irrational, then $x + y$ is irrational.

Proof. The sentence is of the type

$$\forall x \forall y, (P \wedge Q) \rightarrow R$$

where

P: x is rational

Q: y is irrational

R: $x + y$ is irrational

For contradiction, assume

$$\exists x \exists y, \sim\left[(P \wedge Q) \rightarrow R\right]$$

or

$$\exists x \exists y, (P \wedge Q) \wedge \sim R.$$

That is, assume there exists an x and y such that

x is rational

y is irrational

$x + y$ is **not** irrational (rational)

Since x and $x + y$ are rational,

$$x = \frac{a}{b}, \text{ for some integers } a \text{ and } b,\ b \neq 0,$$

$$x + y = \frac{c}{d}, \text{ for some integers } c \text{ and } d,\ d \neq 0.$$

Then

$$\begin{aligned}(x+y)-x &= \frac{c}{d}-\frac{a}{b}\\ &= \frac{cb-da}{db}\end{aligned}$$

Since $cb - da$ and db are both integers,

$(x + y) - x$ is a rational number.

But

$(x + y) - x = y$,

so y is rational. That is,

$\sim Q : \sim$ (y is irrational).

Hence we have $Q \wedge \sim Q$, a contradiction. ■

Let us compare three modes of proof for proving sentences of the type $P \rightarrow Q$. Suppose $A_1, \ldots, A_n$ are the axioms and previously proved theorems.

RCP:

$A_1, \ldots, A_n, P \vdash Q$

Contrapositive:

$A_1, \ldots, A_n, \sim Q \vdash \sim P$

Contradiction:

$$A_1, \ldots, A_n, P, \sim Q \vdash R \wedge \sim R$$

EXAMPLES. These are examples of contradiction proofs.

$$A_1, \ldots, A_n, P, \sim Q \ \vdash P \wedge \sim P$$
$$A_1, \ldots, A_n, P, \sim Q \ \vdash Q \wedge \sim Q$$
$$A_1, \ldots, A_n, P, \sim Q \ \vdash A_i \wedge \sim A_i,\ A_i \in \{A_1, \ldots, A_n\}$$

Comparing, we see that with RCP we assume P with the explicit intention of deducing Q. With the contrapositive, we assume $\sim Q$ with the explicit intention of deducing $\sim P$. But in using proof by contradiction, we assume *both* P and $\sim Q$ and try to deduce *any* sentence R and its negation $\sim R$. Now $\sim R$ could be $\sim P$, Q, $\sim A_i$ for some known fact A_i, or could be some sentence and its negation deduced from $A_i, \ldots, A_n, P,$ and $\sim Q$.

Proof by contrapositive and proof by contradiction with conclusion $\sim P$ are similar; but a proof by contradiction assumes P, while proof by contrapositive does not. For example, in the proof of EXAMPLE 1, the contradiction is of a previously-known fact. In the proof of EXAMPLE 2, the contradiction is of a constituent part of the sentence.

EXAMPLE 3. Suppose f is a function.

<u>Prove</u>: If for every $p > 0$ and every x, $f(x + p) = f(x)$, then f is constant. (1)

Proof.

a) Translate sentence (1) to logical symbolism.

$\left[\forall p > 0\ \forall x,\ f(x+p) = f(x)\right] \to f$ is constant

b) Form the negation of sentence (1).

$\left[\forall p > 0\ \forall x,\ f(x+p) = f(x)\right] \wedge f$ is *not* constant.

For contradiction, assume the negation of (1). Now f is not constant iff $\exists x \exists y,\ f(x) \neq f(y)$. Thus there exists an x and y such that

$$f(x) \neq f(y).$$

Now

$$x = y \rightarrow f(x) = f(y)$$

is always true for functions. By contrapositive it follows that

$$f(x) \neq f(y) \rightarrow x \neq y.$$

Therefore, since $x \neq y$,

$$x < y \text{ or } y < x.$$

CASE 1) $x < y$.

Then by P1 there exists a $p' > 0$ such that

$$x + p' = y.$$

So

$$f(x + p') = f(y).$$

But since $p' > 0,$ by the negation of (1) it follows that

$$f(x + p') = f(x)$$

Hence $f(x) = f(y)$. Thus, we have the contradiction

$$f(x) = f(y) \wedge f(x) \neq f(y),$$

which was deduced from the negation of (1).

CASE 2. $y < x$.

Similar to CASE 1.

Since we deduced a contradiction in both cases we have proved (1). That is, from the negated sentence we deduced an "or" sentence, and from it we deduced a contradiction. ■

Exercise Set 2.8

Prove each of the following by contradiction. At the outset translate each sentence and negate it. Note carefully the deduced contradiction. The proofs in Exercises 1-9 refer to the real number system.

1. For every nonzero x and every y, if x is rational and y is irrational, then $x \cdot y$ is irrational.

2. For every x and every y, $\left(x \neq 0 \wedge y \neq 0\right) \rightarrow xy \neq 0$

3. For every x and every y, $x > 0 \rightarrow x^{-1} > 0$

4. For every x and every y, $x < 0 \rightarrow x^{-1} < 0$

5. For every $x > 0$, $\sqrt{x} < \sqrt{x+1}$ (*Hint:* Use the fact that for every x, $x < x + 1$.)

6. For every $x > 0$, $x + x^{-1} \geq 2$

7. a) There exists an irrational number a and an irrational number b such that a^b is rational.

   b) Does this proof by contradiction actually exhibit an a and b such that a^b is rational?

8. In the system of real numbers the sentence

   $$\forall x,\ x + 0 = 0 + x = x$$

   is true. Suppose there is another real number k such that

   $$k \neq 0 \text{ and } \forall x,\ x + k = k + x = x.$$

   Deduce a contradiction.

9. In the system of real numbers the sentence

   $$\forall x,\ x \cdot 1 = 1 \cdot x = x$$

   is true. Suppose there is another real number k such that

   $$k \neq 1 \text{ and } \forall x,\ x \cdot k = k \cdot x = x.$$

   Deduce a contradiction.

The proofs in Exercises 10-15 refer to the integers. Even though we have given a proof by contrapositive earlier for some, give a contradiction proof.

10. For every x, if x^2 is even, then x is even.

11. For every x, if x^2 is odd, then x is odd.

12. For every x, if x is even, then $x + 1$ is odd.

13. For every x, if x is odd, then $x + 1$ is even.

14. For every $x > 0$ there exists an even number m such that $m > x$.

15. For every $x > 0$ there exists an odd number m such that $m > x$.

The following proof refers to the real number system.

16. For every positive number a and every positive number b,

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}.$$

2.9 PROOFS OF EXISTENCE AND UNIQUENESS

The sentence

There exists an x such that $P(x)$ (1)

is symbolized

$$\exists x P(x).$$

The sentence

There exists exactly one x such that $P(x)$ (2)

is symbolized

$$\exists ! x P(x).$$

Other sentences which have the same meaning as (2) are

There exists a unique x such that $P(x)$;

There exists at least one x such that $P(x)$, and there exists at most one x such that $P(x)$;

There exists one and only one x such that $P(x)$.

Proving Sentences of the Type $\exists ! x P(x)$.

There are two parts to a proof of $\exists ! x P(x)$.

a) **Existence Part.** Proving

$$\exists x P(x),$$

that is, prove there is an x such that $P(x)$ is true.

b) **Uniqueness Part.** Here we must prove that if there are two elements x and z such that

$P(x)$ is true and $P(z)$ is true

then they must be equal. Thus, we must prove

$$\forall x \forall z, \left[P(x) \wedge P(z) \right] \rightarrow x = z.$$

EXAMPLE. Prove: There is a unique x such that for every y, $x + y = y + x = y$ in the real number system.

Translating to logical symbolism we see that we must prove

$$\exists! x \forall y, x + y = y + x = y$$

Proof.

a) **Existence Part.** Prove

$$\exists x P(x)$$

where $P(x)$ is $\forall y, x + y = y + x = y$. Since

$$\forall y, 0 + y = y + 0 = y$$

we know that 0 is such an x.

b) **Uniqueness Part.** Prove

$$\forall x \forall z, \left[P(x) \wedge P(z)\right] \to x = z$$

Let x and z be arbitrary and assume $P(x) \wedge P(z)$ true. Then

$$\forall y, x + y = y + x = y \qquad (1)$$

and

$$\forall y, z + y = y + z = y. \qquad (2)$$

Now from (1) we can substitute z for y and get

$$x + z = z + x = z$$

Similarly from (2) we can substitute x for y and get

$$z + x = x + z = x.$$

Therefore

$$x = z. \blacksquare$$

We could have done a proof by contradiction to complete the uniqueness part of the proof:

From the existence part, we know that 0 is such a number. Now suppose there is another number k such that for every x,

$$x + k = k + x = x$$

and such that $k \neq 0$. Then since this holds for every x, it holds for 0. That is,

$$0 + k = k + 0 = 0.$$

But we also know that $k + 0 = k$. Thus $k = 0$, a contradiction. ■

Exercise Set 2.9

Translate the following to logical symbolism.

1. There is a unique line ℓ such that $P \in \ell$ and $Q \in \ell$.
2. There is exactly one line containing points P and Q.
3. There is exactly one x such that for every y, $x + y = y + x = y$.
4. There is one and only one x such that for every y, $x \cdot y = y \cdot x = y$.
5. For every x and every y there is a unique z such that $x + y = z$.
6. For every x there is a unique y such that $x + y = y + x = 0$.
7. For every x there is a unique y such that if $x \neq 0$, then $x \cdot y = 1$.

Prove:

8. There is exactly one x such that for every y, $x \cdot y = y \cdot x = y$.
9. For every x there is a unique y such that $x + y = y + x = 0$.
10. For every x there is a unique y such that if $x \neq 0$, then $x \cdot y = y \cdot x = 1$.

2.10 PROOF CREATIVITY

In the preceding part of this chapter you learned several modes of proof. The intent is that these will become part of you as tools of proof in the same way that brushes and paints become part of an artist's tools. Now just because the artist has the tools does not guarantee that he will be able to create a painting. Similarly, knowing the modes of proof does not guarantee that you will be able to create a proof; but there are some helpful procedures to follow as aids in proof creativity. We consider these now.

Translate to Logical Symbolism. A typical comment made when proofs are attempted is "I do not know where to start!" One procedure to follow is comparable to that for solving a problem in basic algebra.

Compare:

Algebra Problem: The length of a rectangle is 3 ft more than the width and the area is 54 ft^2. Find its dimensions.

STRATEGY:

1) **Translate to an equation**:

$w(w + 3) = 54$

2) **Examine the equation, select a method of solution from**:

a) Factoring

b) Completing the square

c) Using the quadratic formula

Proof Problem: Every sum of a rational number and an irrational number is irrational. Prove this.

STRATEGY:

1) **Translate to Logical Symbolism**:

$$\forall x \forall y, \left(x \text{ rat. } \wedge\ y \text{ irrat.}\right) \rightarrow x + y \text{ irrat.}$$

2) Examine the translated sentence; select a mode of proof from:

a) Rule of Conditional Proof

b) Contrapositive

c) Contradiction

To solve an algebra problem, we can translate to an equation. Then seeing the structure of the equation, we can select a method of solution. Similarly, to create a proof, we can translate to logical symbolism. Then seeing the structure of the translated sentence, we can select a mode of proof. In algebra you studied methods of solving equations before you attempted applied problems. Here we have studied modes of proof first. In the remainder of the book and in any future mathematics you study, you will use these modes of proof.

Knowing a mode of proof that could be used may still not guarantee success. For example, suppose you wanted to attempt to prove a sentence of the type $P \rightarrow Q$ by using the

Rule of Conditional Proof. You want to assume P and deduce Q. A question often asked is "How do I get from P to Q?" There is no royal road to success; certainly, knowing to assume P and deduce Q is a step in the right direction. The mode of proof provides the structure for the proof; building this structure is usually a more creative task. The following are a few procedures helpful in carrying out modes of proof.

Analogy. In a previous example, we gave a proof of

a is even → a^2 is even.

In the exercises which followed you proved

a is odd → a^2 is odd.

Did you notice that the proofs were analogous? That is, the proof in the example should have suggested a way of proving the sentence in the exercise.

Thus, an important aid in carrying out proofs is to get ideas from other proofs. This is supported by comments of mathematicians who argue that to be good at mathematics you need lots of practice and lots of exposure to different proofs.

Analytic Process. (Working backwards) You want to prove $P \rightarrow Q$. Start with Q and try to find an R such that $R \rightarrow Q$. Then try to find an S such that $S \rightarrow R$. Then you might discover that $P \rightarrow S$.

Assume: P

Deduce: Q

Analytic Process: Q if R ($R \rightarrow Q$)

R if S ($S \rightarrow R$)

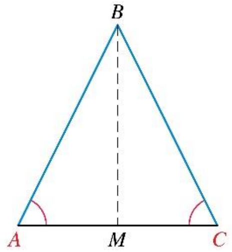
S if P ($P \rightarrow S$)

Hence $P \longrightarrow S \longrightarrow R \longrightarrow Q$

$\therefore P \rightarrow Q$

When reading a proof of $P \rightarrow Q$ in a text one may only see the sequence $P \longrightarrow S \longrightarrow R \longrightarrow Q$ and be amazed at how the author came about it. If the analytic process were used, it probably was not mentioned.

EXAMPLE. Consider $\Delta ABC.$



<u>Prove</u>: $\overline{AB} \cong \overline{CB} \to \angle A \cong \angle C.$

Proof. Assume: $\overline{AB} \cong \overline{CB}$

Deduce: $\angle A \cong \angle C$

Consider $\angle A$ and $\angle C.$ These angles would be congruent *if* they were corresponding angles of congruent triangles. You might think of drawing a segment from *B* to the midpoint *M* of $\overline{AC}.$ Then analytically,

$\angle A \cong \angle C$ if $\angle AMB \cong \angle CMB$ (SSS)

$\vartriangle AMB \cong \vartriangle CMB$ if a) $\overline{BM} \cong \overline{BM}$ (Same segment)

b) $\overline{AM} \cong \overline{CM}$ ($\overline{BM}$ bisects $\overline{AC}$)

c) $\overline{AB} \cong \overline{CM}$ (Assumed)

Then the proof would begin with steps (c), (b), and (a); deduce $\Delta AMB \cong \Delta CMB$ and $\angle A \cong \angle C.$ ■

Starting with the Conclusion. Care should be taken to note that the use of the Analytic Process is logically valid. The process we are about to describe may not be logically valid, but could lead to a valid argument (a proof).

Suppose you want to prove $P \to Q.$ Start with *Q* and deduce as much as you can from it. For example, you might prove $Q \to X \to S \to P,$ and then try to retrace your steps. Now this is not a proof because $(Q \to P) \to (P \to Q)$ is *not* a tautology.

Though the preceding was not a proof you could try to retrace the steps to see if the implications could be turned around:

$$\begin{array}{ccccccc} Q & \to & X & \to & S & \to & P \\ & \leftarrow & & \leftarrow & & \leftarrow & \\ & ? & & ? & & ? & \end{array}$$

If so, then

$$P \to S \to X \to Q$$
$$\therefore P \to Q$$

and you have a proof.

EXAMPLE 1. Given $f(x) = 2x + 1$ and $\varepsilon > 0$. Find a $\delta > 0$ such that $|x| < \delta \to |f(x) - 1| < \varepsilon.$

Start with

$$|f(x) - 1| < \varepsilon.$$

Then

$$\begin{aligned} |f(x)-1| < \varepsilon &\to |(2x+1)-1| < \varepsilon \\ &\to |2x| < \varepsilon \\ &\to -\varepsilon < 2x < \varepsilon \\ &\to -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2} \\ &\to |x| < \frac{\varepsilon}{2} \end{aligned}$$

Checking backwards we see that each arrow could be reversed; that is, for $\delta = \varepsilon/2,$

$$|x| < \delta \to |f(x) - 1| < \varepsilon. \tag{1}$$

Often in such a proof an author will say “Let $\delta = \varepsilon/2$” and then prove (1). What puzzles one is how he knew to let $\delta = \varepsilon/2.$ The author knew because he had started with the conclusion and proceeded as above. This method works when the implications are actually biconditionals.

EXAMPLE 2. Obtaining extraneous solutions. To solve

$$5 + \sqrt{x+7} = x \ \ (x \in \mathbb{R}),$$

we could assume x is a number such that $5 + \sqrt{x+7} = x.$ Note that what we are looking for is the solution, but we start by assuming we already have it. Then

$$\begin{aligned}5+\sqrt{x+7}=x\rightarrow\sqrt{x+7}&=x-5\\ &\rightarrow x+7=\left(x-5\right)^2\\ &\rightarrow x^2-11x+18=0\\ &\rightarrow\left(x-9\right)\left(x-2\right)=0\\ &\rightarrow x=9\vee x=2\end{aligned}$$

What we have shown is

$$5+\sqrt{x+7}=x\rightarrow x=9\vee x=2.$$

We must determine the truth of

$$x=9\vee x=2\rightarrow 5+\sqrt{x+7}=x.$$

Checking,

For 9: $5+\sqrt{9+7}=5+4=9,$ (True)

For 2: $5+\sqrt{2+7}=5+3=8.$ (False)

We see that

if $x = 9$ then $5+\sqrt{x+7}=x.$

Invalid reasoning would have produced the extraneous solution $x = 2$.

Remember to *watch your logic* when you start with the conclusion!

Do-Something Approach (Trial and Error). You want to prove $P \rightarrow Q$ by assuming P and deducing Q. You have no particular way to get from P to Q; but start out, get involved, do something, try different approaches, prove all you can. You might happen onto the proof. This could be illustrated as follows:

$P \rightarrow R \rightarrow T \rightarrow S$?,

$P \rightarrow M \rightarrow Y \rightarrow Z \rightarrow V$?,

$P \rightarrow W \rightarrow X \rightarrow Q$ success!

The do-something approach can also be used with the modes of proof. You try RCP and get nowhere. Maybe, you can prove the contrapositive.

When reading proofs in mathematics texts and journals, one is not aware of the blind alleys and unsuccessful attempts preceding a successful proof. This leads one to think the established mathematician never follows a blind alley or makes a mistake. Trial and error is very much a part of mathematical creativity.

Use of Definitions. Another helpful procedure is to recall all relevant definitions. It is a tendency to read a definition and ignore its importance in later proofs. To illustrate, suppose the following definition is given in set theory.

Definition. For any two sets A and B,

$A \subseteq B$ iff for every x, $x \in A \to x \in B$.

Later the following theorem is to be proved:

Theorem. For any two sets A and B,

$A \cap B \subseteq A$.

To accomplish the proof one would let A and B be arbitrary sets and prove $A \cap B \subseteq A$. A stumbling block may be met at this point *unless one uses the definition* to interpret what it means for $A \cap B$ to be contained in A. That is, using the definition it follows that one must prove:

For every x, $x \in A \cap B \to x \in A$.

Use of Previously-Proved Theorems. It is also helpful in starting a proof to examine previously-proved theorems for results which might be relevant to the proof.

Let us summarize some strategies for proof creativity:

(1) Translate to logical symbolism.

(2) Examine the translated sentence; select a mode of proof.

(3) After a reasonable effort with one mode of proof, try another.

(4) Examine analogous proofs for hints.

(5) Use the definitions.

(6) Use the results of previous theorems.

(7) Realize that trial and error are very much a part of proof creativity.

As you continue in mathematics, it may be of help to read and re-read this section.

Exercise Set 2.10

1. Describe how the analytic process could be used to prove the contrapositive of $P \to Q$.

2. Describe how the analytic process could be used to prove $P \to \sim Q$.

For Exercises 3-10, assume the following are proved:

a) $A \cup B = B \cup A$

b) $(A \cup B) \cup C = A \cup (B \cup C)$

c) $x \in \mathbb{Q} \wedge y \in J \to x + y \in J$

d) The solution of $5+\sqrt{x+7} = x$ is $x = 9$.

e) $A \subseteq B \to B' \subseteq A'$

f) $\int (f+g) = \int f + \int g$

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

h) $(A \cup B)' = A' \cap B'$

Suppose that you wanted to prove the sentences in Exercises 3 through 10. State an analogous proof for an above sentence you might examine for a hint.

3. $B \subseteq A' \to A \subseteq B'$

4. $\left|\int (f+g)\right| \leq \int |f| + \int |g|$

5. $A \cap B = B \cap A$

6. There is a solution for $\sqrt{2x-1} = x-2$

7. $x \in \mathbb{Q} \wedge y \in J \to x - y \in J$

8. $(A \cap B) \cap C = A \cap (B \cap C)$

9. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

10. $(A \cap B)' = A' \cup B'$

11. Find the solution for Exercise 6.

12. Given $f(x) = 3x + 2$ and $\varepsilon > 0$. Find a $\delta > 0$ such that $|x| < \delta \to |f(x) - 2| < \varepsilon$.

CHAPTER 3 SETS

3.1 BASIC SET PROPERTIES

This chapter serves both as practice of the previous chapters and as a preparation for advanced courses.

Underlying every mathematical system is the theory of sets. In this chapter we prove many properties of sets. We will be considering set theory less formally than it could be studied. Such a study belongs to more advanced courses.

The notion of *number* is undefined in mathematics. Nevertheless, we have an idea of what a number is. For example, we get the idea of 3 by thinking of three objects. Similarly, the notions of *sets* and *elements* of sets are undefined even though we have an idea of what they are.

We consider a universal set U. (The word *universal* is also undefined.) Recall that

$a \in A$ means a "is an element of" A ($\in$ is an undefined relation between the elements of sets and sets themselves).pa

AXIOM 1 (Equality). $A = B \leftrightarrow \forall x, x \in A \leftrightarrow x \in B$

AXIOM 2. a) $A = U \leftrightarrow \forall x, x \in A,$

b) $\forall x, x \in U.$

Axiom 2 asserts that every element is in the universal set.

AXIOM 3. a) $A = \varnothing \leftrightarrow \forall x, x \notin A,$

b) $\forall x, x \notin \varnothing.$

Axiom 3 asserts that no element is in the empty set.

AXIOM 4. $U \neq \varnothing$

Axiom 4 restates what we mentioned in Chapter 1; we only consider non-empty universal sets.

AXIOM 5. $A \notin A, x \neq \{x\}$

Axiom 5 and Axiom 6 which follow attempt to eliminate some difficulties with sets that arose when Georg Cantor (1845-1918) developed a set theory between 1874 and 1884.

EXAMPLES. Each of the following is false.

$\{a,b\} \in \{a,b\}, \varnothing \in \varnothing,$

$\varnothing = \{\varnothing\}, 3 = \{3\}.$

Let us compare 3 and {3}. Why do we know 3 ≠ {3}? We could justify this by Axiom 5. But intuitively the idea of 3 is a number and the idea of {3} is a set. The number 3 is abstracted out of sets of three objects. But {3} is a set with one object, the number 3. Thus, the ideas are different.

Let us compare $\varnothing$ and $\{\varnothing\}$. Again $\varnothing \neq \{\varnothing\}$ by Axiom 5, but let us think intuitively. The symbol $\varnothing$ stands for the empty set. It has nothing in it. The symbol $\{\varnothing\}$ stands for a set which has one element, the empty set itself. You might think of $\varnothing$ as an empty box and $\{\varnothing\}$ as an empty box in a box. The two notions are different.

Definition 1. $A \subseteq B \leftrightarrow \forall x, x \in A \to x \in B$

EXAMPLES. Each of the following is true.

$3 \in \{3\}, \{3\} \subseteq \{3\}, \{3\} = \{3\}, \{3\} \subseteq \{3,4\}, \varnothing = \varnothing, \varnothing \subseteq \varnothing, \varnothing \subseteq \{\varnothing\}, \varnothing \in \{\varnothing\}$

Note that $\varnothing$ is both an element of and a subset of $\{\varnothing\}$. That is, it is true that

$$\varnothing \in \{\varnothing\} \text{ and } \varnothing \subseteq \{\varnothing\},$$

but the following are false:

$$\{3\} \in \{3\} \text{ and } 3 \subseteq \{3\}.$$

The first is false by Axiom 5. The second is false because 3 is not a set.

AXIOM 6. If for every $x \in U$, $P(x)$ is a statement about x, then there exists a set B such that

a) $x \in B \leftrightarrow P(x)$ is true

and b) $B = \{x | x \in U \wedge P(x)\}$.

Note that Axiom 6 asserts the existence of a set defined by an open sentence whose variable represents elements of U. It is possible that the set thus described is actually the empty set.

Definition 2. a) $A \cup B = \{x | x \in A \vee x \in B\}$

b) $x \in A \cup B \leftrightarrow x \in A \vee x \in B$

Definition 3. a) $A \cap B = \{x | x \in A \wedge x \in B\}$

b) $x \in A \cap B \leftrightarrow x \in A \wedge x \in B$

Two sets are ***disjoint*** iff $A \cap B = \varnothing$.

Definition 4. a) $A' = \{x | x \notin A\} = \{x | x \in U \wedge x \notin A\}$

b) $x \in A' \leftrightarrow x \notin A$

Part b) of each of Definitions 2-4 follows by application of Axiom 6, and, though somewhat redundant, is stated for convenience.

Now we prove some theorems. You should make use of the logic and modes of proof previously considered. In many proofs, we mention the mode(s) of proof used. When a mode is not mentioned, you should decide what is used. The ANALYSIS notes, listed at the end of some of the proofs, explain how the proofs might have been created. Of importance is to note how closely the logic results relate to many of the proofs.

Theorem 1. $A = \varnothing \leftrightarrow \sim \exists x, x \in A$

Proof. **Mode of Proof:** Iff-string.

$$\begin{aligned} A = \varnothing &\leftrightarrow \forall x,\ x \notin A, \text{ by Axiom 3} \\ &\leftrightarrow \sim \exists x,\ x \in A, \text{ by properties of negation.} \end{aligned}$$

■

ANALYSIS: Recollection of rules of negation.

Theorem 2. For any subset A of U, $\varnothing \subseteq A$.

Proof. Let A be an arbitrary subset. We must prove $\varnothing \subseteq A$. How would we prove this? Remember to use definitions. By Definition 1, we must prove $\forall x, x \in \varnothing \to x \in A$. By Axiom 3, we know $\forall x, x \notin \varnothing$. That is for every $x \in U$, $x \in \varnothing$ is false. Thus, the conditional

$$x \in \varnothing \to x \in A$$

always has a *false* antecedent, and is always true. Hence $\varnothing \subseteq A$. ■

Theorem 3. $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$

Proof. **Mode of Proof:** Iff-string.

$$\begin{aligned} A = B &\leftrightarrow \forall x,\ x \in A \leftrightarrow x \in B, \text{ by Axiom 1} \\ &\leftrightarrow \forall x\left[\left(x \in A \to x \in B\right) \wedge \left(x \in B \to x \in A\right)\right], \text{ definition of } \leftrightarrow \\ &\leftrightarrow \left(\forall x, x \in A \to x \in B\right) \wedge \left(\forall x, x \in B \to x \in A\right), \text{ by the rule of reasoning} \\ &\qquad \forall x\left[P\left(x\right) \wedge Q\left(x\right)\right] \leftrightarrow \left[\forall x P\left(x\right) \wedge \forall x Q\left(x\right)\right] \\ &\leftrightarrow A \subseteq B \wedge B \subseteq A, \text{ by Definition 1.} \end{aligned}$$

■

Theorem 3 provides a very useful way of proving two sets equal.

Theorem 4. a) $A \cap B = B \cap A$

b) $A \cup B = B \cup A$

Proof.

a) We use Axiom 1 to prove this. That is, we prove $\forall x, x \in A \cap B \leftrightarrow x \in B \cap A$. Let x be arbitrary. Then

$$\begin{aligned} x \in A \cap B &\leftrightarrow x \in A \wedge x \in B, \text{ by Definition 3} \\ &\leftrightarrow x \in B \wedge x \in A, \text{ by the tautology } P \wedge Q \leftrightarrow Q \wedge P \\ &\leftrightarrow x \in B \cap A, \text{ by Definition 3.} \end{aligned}$$

■

ANALYSIS: The analogy between $A \cap B = B \cap A$ and the tautology $P \wedge Q \leftrightarrow Q \wedge P$ provided the idea for the proof.

b) Left as an exercise.

Theorem 5. $(A')' = A$

Creating the Proof: Can you think of an analogous tautology which might help?

Proof. Let *x* be arbitrary.

$$\begin{aligned} x \in (A')' &\leftrightarrow x \notin A', \text{ by Definition 4} \\ &\leftrightarrow \sim(x \in A'), \text{ by definition of the negation symbol} \\ &\leftrightarrow \sim(x \notin A), \text{ by Definition 4} \\ &\leftrightarrow \sim\sim(x \in A), \text{ by definition of the negation symbol} \\ &\leftrightarrow x \in A, \text{ by the tautology } \sim\sim P \leftrightarrow P. \end{aligned}$$

■

Theorem 6. $A \subseteq B \leftrightarrow A \cap B' = \varnothing$

Proof. **Mode of Proof:** Iff-string.

$$\begin{aligned} A \subseteq B &\leftrightarrow \forall x, x \in A \to x \in B, \text{ by Definition 1} \\ &\leftrightarrow \sim \exists x \sim (x \in A \to x \in B), \text{ by a rule of reasoning for negation} \\ &\leftrightarrow \sim \exists x, x \in A \wedge x \notin B, \text{ by the tautology } \sim(P \to Q) \leftrightarrow (P \wedge \sim Q) \\ &\leftrightarrow \sim \exists x, x \in A \wedge x \in B', \text{ by Definition 4} \\ &\leftrightarrow \sim \exists x, x \in A \cap B', \text{ by Definition 3} \\ &\leftrightarrow A \cap B' = \varnothing, \text{by Theorem 1.} \end{aligned}$$

■

Theorem 7. $A = B \leftrightarrow A' = B'$

Proof. Left as an exercise.

Theorem 8. $(A \cup B)' = A' \cap B'$

Proof. We must, by Axiom 1, prove

$$\forall x, x \in (A \cup B)' \leftrightarrow x \in A' \cap B'$$

Let *x* be arbitrary.

$$\begin{aligned} x \in (A \cup B)' &\leftrightarrow x \notin (A \cup B), \text{ by Definition 4} \\ &\leftrightarrow \sim (x \in A \cup B), \text{ by definition of the negation symbol} \\ &\leftrightarrow \sim (x \in A \vee x \in B), \text{ by Definition 2} \\ &\leftrightarrow x \notin A \wedge x \notin B, \text{ by the tautology } (\sim P \wedge \sim Q) \leftrightarrow \sim (P \vee Q) \\ &\leftrightarrow x \in A' \wedge x \in B', \text{ by Definition 4} \\ &\leftrightarrow x \in A' \cap B', \text{ by Definition 3.} \end{aligned}$$

■

Theorem 9. $(A \cap B)' = A' \cup B'$

Proof. Left as an exercise.

Theorem 10. a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof.

a) Let *x* be an arbitrary element of *U*. Then

$$\begin{aligned} x \in A \cap (B \cup C) &\leftrightarrow x \in A \wedge x \in (B \cup C), \text{ by Definition 3} \\ &\leftrightarrow x \in A \wedge (x \in B \vee x \in C), \text{ by Definition 2} \\ &\leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C), \text{ by the tautology} \\ &\qquad P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R) \\ &\leftrightarrow x \in (A \cap B) \cup (A \cap C), \text{ by Definitions 2 and 3.} \end{aligned}$$

■

ANALYSIS: Note the similarity between $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and the tautology $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$.

b) Left as an exercise.

Theorem 11. a) $A \subseteq A \cup B$

b) $A \cap B \subseteq A$

Proof.

a)

$$\begin{aligned} x \in A &\to x \in A \vee x \in B, \text{ by the tautology } P \to (P \vee Q) \\ &\to x \in A \cup B, \text{ by Definition 2.} \end{aligned}$$

■

b) Left as an exercise.

Theorem 12. $A \subseteq B \to B' \subseteq A'$

Proof.

$$\begin{aligned} A \subseteq B &\to \forall x (x \in A \to x \in B), \text{ by Definition 1} \\ &\to \forall x (x \notin B \to x \notin A), \text{ using the contrapositive of } x \in A \to x \in B \\ &\to \forall x (x \in B' \to x \in A'), \text{ by Definition 4} \\ &\to B' \subseteq A', \text{ by Definition 1} \end{aligned}$$

Hence $A \subseteq B \to B' \subseteq A'$ for every two subsets *A* and *B* of *U*. ■

ANALYSIS: Note the similarity between
$A \subseteq B \to B' \subseteq A'$ and the tautology $(P \to Q) \to (\sim Q \to \sim P)$.

Exercise Set 3.1

When doing these proofs note what mode(s) of proof you are using. Also, include your own analysis notes; that is, what provided the bud of the idea for the proof. For example, did you use a proof analogous to a previous one, did you use the do-something approach, and so on? You may use any previous theorem or exercise when doing a proof. Compare your proof with that of the answer key; they may differ.

Let *A*, *B*, and *C* be arbitrary subsets of *U*. Prove the following.

1. Theorem 4-b)

2. Theorem 7

3. Theorem 9

4. Theorem 9, but give a proof which uses Theorems 5, 7, and 8

5. Theorem 10-b)

6. Theorem 11-b)

7. a) $A \subseteq A$ b) $A = A$

8. $\varnothing' = U$

9. $A \subseteq B$ iff $A' \cup B = U$

10. $A \cup (B \cup C) = (A \cup B) \cup C$

11. $A \cap (B \cap C) = (A \cap B) \cap C$

12. $A \cup \varnothing = A$ *Hint:* Use Theorems 2 and 11.

13. $A \cup A' = U$ *Hint:* $P \vee \sim P$ is a tautology.

14. $A \cap B \subset A \cup B$

15. $\varnothing = U'$

16. $A \cap \varnothing = \varnothing$ *Hint:* Theorems 2, 3, and 11.

17. $A \subseteq U$

18. a) $A \cap U = A$ b) $A \cup U = U$

19. $A \cap A' = \varnothing$

20. $A \subseteq B' \rightarrow B \subseteq A'$

21. $A \cup A = A$

22. $A \cap A = A$

23. $A \subseteq B \leftrightarrow A \cup B = B$ *Hint:* Start with $A \cup B = B$ and try to find a helpful tautology.

24. $A \subseteq B \leftrightarrow A \cap B = A$

25. $A \subseteq \varnothing \leftrightarrow A = \varnothing$

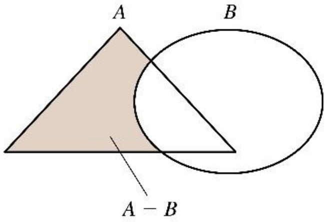
26. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

27. If $A \subseteq B$, then $A \cup C \subseteq B \cup C$

Definition 5.

$$\begin{aligned} A - B &= \{x \mid x \in A \wedge x \notin B\} \\ &= A \cap B' \end{aligned}$$

= the ***difference*** of A and B



28. $A - \varnothing = A$ *Hint:* Use Exercises 2 and 12.

29. $A - A = \varnothing$

30. $A \subseteq B \to B - (B - A) = A$

31. $(A - B) - C = (A - C) - B$

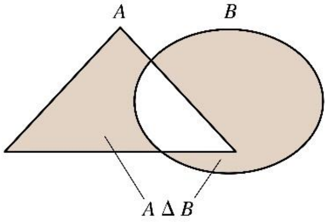
32. $A - (B - C) = A \cap (B' \cup C)$

33. Let ℤ = the set of integers. Show that $\forall A \forall B,\ A - B = B - A$ is false by finding two sets in ℤ for which $A - B \neq B - A$.

Definition 6.

$$\begin{aligned} A \,\Delta\, B &= \{x \mid x \in A \cup B \wedge x \notin A \cap B\} \\ &= (A \cup B) - (A \cap B) \end{aligned}$$

= the ***symmetric difference*** of A and B



34. $A \,\Delta\, B = B \,\Delta\, A$

35. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

36. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

37. $A \Delta B = (A - B) \cup (B - A)$

38. $A \cap B = \varnothing \rightarrow A \Delta B = A \cup B$

39. $A \cup B = (A \Delta B) \Delta (A \cup B)$

3.2 MORE SET PROPERTIES

Power Set.

Definition 7. $\wp(A)=$ the ***power set*** of $A = \{B \mid B \subseteq A\}$. Thus, $\wp(A)$, the power set of A, is the set of all subsets of A.

For example, the set $\{a, b\}$ has the following subsets:

$$\varnothing,\ \{a\},\ \{b\},\ \{a, b\}.$$

Thus $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$. Note that $\wp(A)$ is a set whose elements are sets.

Indexed Unions and Intersections. We now consider the problem of defining larger unions and intersections of sets. Consider three sets A_1, A_2, and A_3. We know from previous results that

$$(A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3).$$

This is the ***associative law*** for unions. It says, in effect, that to find the union of three sets we find the union of any two, then form the union of that set with the third set. This says that we could drop the parentheses and use the notation

$$A_1 \cup A_2 \cup A_3.$$

Let us look at this from a logical standpoint.

$x \in A_1 \cup A_2 \cup A_3$ iff $x \in A_1$ or $x \in A_2$ or $x \in A_3$.

That is, there exists an $i \in \{1,2,3\}$ such that $x \in A_i$, or x is in the union iff it is in one of the sets.

Similarly, $x \in A_1 \cap A_2 \cap A_3$ iff $x \in A_1$ and $x \in A_2$ and $x \in A_3$. That is, for every $i \in \{1,2,3\}$, $x \in A_i$, or x is in the intersection iff it is in all the sets.

This leads us to the following definition.

Definition 8. For any finite collection of sets $A_1, \ldots, A_n$ when $F = \{1, 2, \ldots, n\}$, we define

$\bigcup_{i=1}^{n} A_i$, or $\bigcup_{i \in F} A_i$, and $\bigcap_{i=1}^{n} A_i$, or $\bigcap_{i \in F} A_i$, as follows:

a) $\bigcup_{i=1}^{n} A_i = \bigcup_{i \in F} A_i = \{x \mid \exists i \in F, \text{ such that } x \in A_i\}$

b) $\bigcap\limits_{i=1}^{n} A_i = \bigcap\limits_{i\in F} A_i = \{x \mid \forall i \in F, x \in A_i\}$

EXAMPLE 1. Suppose

$$A_1 = \{2,11\},\ A_2 = \{3,5,7,11\},\ A_3 = \{4,6,11\},\ \text{and } A_4 = \{5,11\},$$

where $F = \{1,2,3,4\}$. Then

$$\bigcup_{i=1}^{4} A_i = \bigcup_{i\in F} A_i = \{2,3,4,5,6,7,11\},$$
$$\bigcap_{i=1}^{4} A_i = \bigcap_{i\in F} A_i = \{11\}.$$

F is called an ***indexing set*** and $\{A_1, A_2, A_3, A_4\}$, or $\{A_i\}_{i=1}^{n}$, or $\{A_i\}_{i\in F}$, is called a ***family***, or **collection of sets**.

EXAMPLE 2. Suppose

$$A_1 = \{x \mid 1 \le x \le 1+1\},\ A_2 = \{x \mid 1 \le x \le 1+\tfrac{1}{2}\}, \ldots,\ A_n = \{x \mid 1 \le x \le 1+\tfrac{1}{n}\},$$

when $F = \{1,2,\ldots,n\}$.

Then

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i\in F} A_i = \{x \mid 1 \le x \le 1+\tfrac{1}{n}\} = A_n,$$
$$\bigcup_{i=1}^{n} A_i = \bigcup_{i\in F} A_i = \{x \mid 1 \le x \le 1+1\} = A_1.$$

Theorem 13. For any finite family of sets $\{A_i\}_{i=1}^{n+1}$,

a) $\bigcup\limits_{i=1}^{n+1} A_i = \left(\bigcup\limits_{i=1}^{n} A_i\right) \cup A_{n+1}$

b) $\bigcap\limits_{i=1}^{n+1} A_i = \left(\bigcap\limits_{i=1}^{n} A_i\right) \cap A_{n+1}$

Proof of a). We use Theorem 3 to do the proof. We prove each set is a subset of the other.

Let $x \in \bigcup\limits_{i=1}^{n+1} A_i$. Then there exists an $i \in \{1,2,\ldots,n,n+1\}$ such that $x \in A_i$. Since

$\{1, 2, \ldots, n, n+1\} = \{1, 2, \ldots, n\} \cup \{n+1\}$, it follows that $x \in A_i$, for $i \in \{1, 2, \ldots, n\}$, in which case $x \in \bigcup_{i=1}^{n} A_i$; or $x \in A_{n+1}$. Thus $x \in \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}$.

Let $x \in \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}$. The proof that $x \in \bigcup_{i=1}^{n+1} A_i$ is left as an exercise. ■

Note that Theorem 13 looks very much like a recursive definition. In fact, we could have taken Theorem 13 as a definition and proved Definition 8 as a Theorem. We used Definition 8 because it extends nicely to the case where we are finding unions and intersections of infinite families of sets. Before we do this let us use Theorem 13 to prove another property of sets. It is a more general case of $(A \cup B)' = A' \cap B'$, proved earlier.

Theorem 14. For any finite family of sets $A_1, \ldots, A_n$,

$$\left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i'$$

(The complement of the union is the intersection of complements.)

Proof. **Mode of Proof:** Mathematical induction, where

$$P(n): \left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i'$$

1) BASIS STEP. Prove $P(1)$: $(A_1)' = A_1'$. This is clear.

2) INDUCTION STEP. Prove: $\forall k\left[P(k) \rightarrow P(k+1)\right]$.

Assume $P(k)$: $\left(\bigcup_{i=1}^{k} A_i\right)' = \bigcap_{i=1}^{k} A_i'$

Deduce $P(k + 1)$: $\left(\bigcup_{i=1}^{k+1} A_i\right)' = \bigcap_{i=1}^{k+1} A_i'$

$$\text{Now } \left(\bigcup_{i=1}^{k+1} A_i\right)' = \left[\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right]', \text{ by Theorem 13}$$

$$= \left(\bigcup_{i=1}^{k} A_i\right)' \cap A_{k+1}', \text{ by } (A \cup B)' = A' \cap B', \text{ where we consider}$$

$$\bigcup_{i=1}^{k} A_i \text{ to be one set and } A_{k+1} \text{ to be the other}$$

$$= \left(\bigcap_{i=1}^{k} A_i'\right) \cap A_{k+1}', \text{ by } P(k)$$

$$= \bigcap_{i=1}^{k+1} A_i'.$$

■

The following are examples of infinite families of sets.

EXAMPLE 3. For every $i \in \mathbb{N}$, the natural numbers, we have a set A_i as follows:

$A_1 = \{x | -1 \le x \le 1\}, A_2 = \{x | -2 \le x \le 2\}, \ldots, A_n = \{x | -n \le x \le n\}, \ldots$. There is an infinite number of sets in this family.

EXAMPLE 4. For every $r \in \mathbb{R}$, the real numbers, we have a set A_r as follows:

$$A_r = \{x | -r \le x \le r\}.$$

Although we will not prove it here, there are more sets in $\{A_r\}_{r \in \mathbb{R}}$ than there are in $\{A_i\}_{i \in \mathbb{N}}$. We now extend Definition 8.

Definition 9. For any family $\{A_i\}_{i \in F}$ and non-empty indexing set F,

a) $\bigcup_{i \in F} A_i = \{x | \exists i \in F \text{ such that } x \in A_i\}$

b) $\bigcap_{i \in F} A_i = \{x | \forall i \in F, x \in A_i\}$

EXAMPLES

a) For EXAMPLE 3 above:

$\bigcup_{i \in \mathbb{N}} A_i$ (or as usually expressed $\bigcup_{i=1}^{\infty} A_i$) = $\mathbb{R}$, the set of real numbers;

$$\bigcap_{i=1}^{\infty} A_i = A_1 = \{x \mid -1 \le x \le 1\}.$$

b) For EXAMPLE 4 above:

$$\bigcup_{i\in\mathbb{R}} A_i = \mathbb{R}, \quad \bigcap_{i\in\mathbb{R}} A_i = \{0\}.$$

Note the differences.

Exercise Set 3.2

1. Suppose $A_1 = \{2,4,5\}$, $A_2 = \{2,3,5,7\}$, $A_3 = \{1,2,3,5\}$, where $F = \{1,2,3\}$.

   Find:

   a) $\bigcup_{i=1}^{3} A_i$, or $\bigcup_{i\in F} A_i$ b) $\bigcap_{i=1}^{3} A_i$, or $\bigcap_{i\in F} A_i$

2. Suppose

   $A_1 = \{x \mid 0 \le x \le 1\}$, $A_2 = \{x \mid 0 \le x \le \tfrac{1}{2}\}, \ldots, A_n = \{x \mid 0 \le x \le \tfrac{1}{n}\}$ where $F = \{1,2,3,\ldots,n\}$.

   Find:

   a) $\bigcup_{i=1}^{n} A_i$, or $\bigcup_{i\in F} A_i$ b) $\bigcap_{i=1}^{n} A_i$, or $\bigcap_{i\in F} A_i$

3. Prove: For any finite family of sets $\{A_i\}_{i=1}^{n}$,

$$\left(\bigcap_{i=1}^{n} A_i\right)' = \bigcup_{i=1}^{n} A_i'.$$

4. Let $A_1 = \{x \mid 1 \le x\}$, $A_2 = \{x \mid 2 \le x\}, \ldots, A_n = \{x \mid n \le x\}, \ldots$. Find:

   a) $\bigcup_{i=1}^{n} A_i$ b) $\bigcap_{i=1}^{n} A_i$ c) $\bigcup_{i=1}^{\infty} A_i$ d) $\bigcap_{i=1}^{\infty} A_i$

5. For any $r \in \mathbb{R}$, let $A_r = \{x \mid r \le x\}$. Find:

   a) $\bigcup_{r\in\mathbb{R}} A_r$ b) $\bigcap_{r\in\mathbb{R}} A_r$

6. For any $i \in \mathbb{N}$, let $A_i = \{x \mid x \in \mathbb{N} \text{ and } x \text{ is a multiple of } i\}$. Thus $A_2 = \{2, 4, 6, 8, \ldots\}$, $A_3 = \{3, 6, 9, 12, \ldots\}$, and $A_n = \{n, 2n, 3n, \ldots\}$. Find:

   a) $A_2 \cap A_3$ b) $A_4 \cap A_5$ c) $A_5 \cap A_8$ d) $\bigcup_{i=1}^{\infty} A_i$ e) $\bigcap_{i=1}^{\infty} A_i$

7. Prove: For any finite family $\{A_i\}_{i=1}^{n}$, and any set B,

   a) $B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$

   b) $B \cup \left(\bigcap_{i=1}^{n} A_i \right) = \bigcap_{i=1}^{n} (B \cup A_i)$

8. Prove: For any family $\{A_i\}_{i \in F}$ and non-empty indexing set F,

   a) $\left(\bigcup_{i \in F} A_i \right)' = \bigcap_{i \in F} A_i'$

   b) $\left(\bigcap_{i \in F} A_i \right)' = \bigcup_{i \in F} A_i'$

9. Prove: For any family $\{A_i\}_{i \in F}$ and non-empty indexing set F, if $i_0 \in F$, then

$$\bigcap_{i \in F} A_i \subseteq A_{i_0} \subseteq \bigcup_{i \in F} A_i .$$

Definition 10. For any family $\{A_i\}_{i \in F}$ and non-empty indexing set F,

$\{A_i\}_{i \in F}$ is ***disjoint*** $\leftrightarrow \bigcap_{i \in F} A_i = \varnothing$;

$\{A_i\}_{i \in F}$ is ***pairwise disjoint*** $\leftrightarrow i \neq j \to A_i \cap A_j = \varnothing$.

10. Let $A_1 = \{2,3,4\}$, $A_2 = \{3,4,5\}$, and $A_3 = \{6,7,8\}$.

a) Find: $\bigcap_{i=1}^{3} A_i$

b) Is $\{A_i\}_{i=1}^{3}$ disjoint?

c) Find: $A_1 \cap A_2$, $A_2 \cap A_3$, and $A_1 \cap A_3$.

d) Is $\{A_i\}_{i=1}^{3}$ pairwise disjoint?

e) Prove or disprove: $\{A_i\}_{i\in F}$ disjoint → $\{A_i\}_{i\in F}$ pairwise disjoint.

f) Prove or disprove: $\{A_i\}_{i\in F}$ pairwise disjoint → $\{A_i\}_{i\in F}$ disjoint.

Prove the following. *Hint:* Try contrapositive or contradiction.

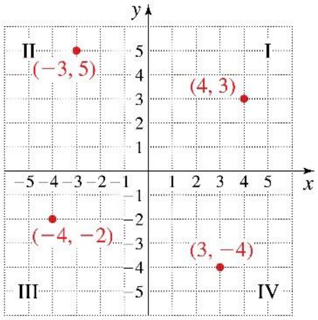
11. For every two subsets A and B of U, $A \cup B \neq \varnothing \rightarrow A \neq \varnothing$ or $B \neq \varnothing$.

12. For every subset A of U, $A \neq A'$.

13. For every two subsets A and B of U, $A \cap B \neq \varnothing \rightarrow A \neq \varnothing$.

3.3 RELATIONS

Ordered Pairs. Let us recall from algebra the meaning of the symbolism (a, b), called an ***ordered pair***. Below is a graph of some ordered pairs. Look at the graph and try to recall some properties of ordered pairs.



Note that none of the pairs are the same. Perhaps you recall the property

$$(a, b) = (c, d) \leftrightarrow a = c \text{ and } b = d.$$

For the pair (a, b), a is called the ***first coordinate*** and b is the ***second coordinate***. The words "first" and "second" suggest the notion of order. By the above property, two ordered pairs are equal if their first coordinates are equal *and* their second coordinates are equal.

We can prove the above property via our set theory by means of a rather strange-looking definition.

Definition 11. $(a,b)=\{\{a\},\{a,b\}\}$

From this we can first prove that, given the elements a and b, the pair (a, b) exists. The sets $\{a\}$ and $\{a,b\}$ exist, thus the set $\{\{a\},\{a,b\}\}$, or (a, b), exists. Note that in the two-element set $\{\{a\}, \{a, b\}\}$, the element $\{a, b\}$ shows the unordered pair, while the element $\{a\}$ shows the first coordinate.

Theorem 15. $(a,b)=(c,d)\leftrightarrow a=c$ and $b=d$

Proof.[*]

a) Prove: $a=c$ and $b=d \rightarrow (a,b)=(c,d)$. Note that when we are trying to prove $(a,b)=(c,d)$ we are proving set equality. From $a = c$ and $b = d$ it follows that $\{a\}=\{c\}$ and $\{a,b\}=\{c,d\}$. Thus $\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$, or $(a,b)=(c,d)$.

---

[*] This proof could be skipped if time is short.

b) Prove: $(a,b)=(c,d) \to a=c$ and $b=d$. From $(a,b)=(c,d)$ it follows that $\{\{a\},\{a,b\}\}=\{\{c\},\{c,d\}\}$. From the definition of set equality, it follows that $(\{a\}=\{c\}$ and $\{a,b\}=\{c,d\})$ or $(\{a\}=\{c,d\}$ and $\{a,b\}=\{c\})$.

CASE 1. $\{a\}=\{c\}$ and $\{a,b\}=\{c,d\}$. Then $a = c$ from $\{a\}=\{c\}$. Thus since $a = c$ and $\{a,b\}=\{c,d\}$ it follows that $b = d$.

CASE 2. $\{a\}=\{c,d\}$ and $\{a,b\}=\{c\}$. From $\{a\}=\{c,d\}$ it follows that there is only one element in $\{c,d\}$, or $a = c = d$. Similarly, from $\{a,b\}=\{c\}$ it follows that $a = b = c$. Thus $a = b = c = d$; in particular, $a = c$ and $b = d$. ■

One could extend Definition 11 to define ordered triples, (a,b,c), and ordered n-tuples $(a_1,\ldots,a_n)$, but we shall omit this for the sake of expediency.

Definition 12. Given two sets A and B, the ***Cartesian product***, or ***cross product***, is defined

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

($A \times B$ is the set of all ordered pairs with first coordinate in A and second coordinate in B.)

EXAMPLE 1. $A=\{1,2,3\}$, $B=\{3,4\}$

$$A \times B = \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$$

$$B \times A = \{(3,1),(3,2),(3,3),(4,1),(4,2),(4,3)\}$$

Note that, in general, $A \times B \neq B \times A$.

EXAMPLE 2. $A=\{1,2\}$

$$A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$$

EXAMPLE 3. $A=\{a,b\}$, $B=\varnothing$

$$A \times B = \varnothing$$

For the set $\mathbb{R}$ of real numbers we can think of $\mathbb{R} \times \mathbb{R}$ as the set of points in a plane.

Relations. Before we define *relation*, let us motivate the notion. Think of the relation "is a parent of"; then the following is a true sentence:

Your *father* is a parent of *you*.

There is a first person, *father,* and a second person, *you*. We can then abstract the notion of an ordered pair:

(father, you)

In fact, there are lots of such ordered pairs if we consider the set of all people.

Think of the relation "less than."

$x<y$	(x,y)
$2<3$	$(2,3)$
$-3<5$	$(-3,5)$
$\frac{1}{2}<\pi$	$(\frac{1}{2},\pi)$

Note that 2 < 3 establishes a "relation" between a *first* number 2 and a *second* number 3. We can again abstract the notion of an ordered pair (2, 3).

Definition 13. Given sets *A* and *B*. A ***relation from A to B*** is a subset of *A* × *B*. That is, ρ (Greek letter *rho*) is a relation from *A* to *B* iff $\rho \subseteq A \times B.$ If *A* = *B*, then ρ is a ***relation on A***.

It is possible to have many relations from a set *A* to a set *B*.

EXAMPLE 4. Let $A=\{1,2,3\}$, $B=\{1,2,3,4,5,6,7,8,9\}$, and $\rho=\{(1,1),(2,4),(3,9)\}.$ Note that $(2,4)\in\rho; (2,4)$ is in the relation $\rho.$ We often write $2\rho 4.$ Sometimes we can describe a relation with a sentence. For the above,

$$\rho=\{(x,y)|x\in A \text{ and } x\in B \text{ and } y=x^2\}.$$

When it is clear what the sets are, we may abuse the notation and refer to the sentence $y=x^2$ as a "relation."

EXAMPLE 5. ℝ = the real numbers.

$$<=\{(x,y)|x\in\mathbb{R},\, y\in\mathbb{R}, \text{ there exists an } m>0 \text{ such that } x+m=y\}$$

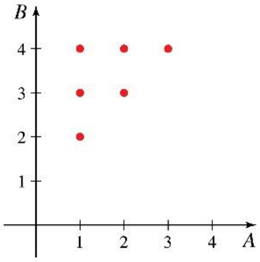
Thus $(2,3)\in<,$ or 2 < 3; $(-5,7)\in<,$ or −5 < 7. The relation < is a ***relation on*** **ℝ**.

We can graph relations as follows. The ***graph*** of a relation is the plot of all ordered pairs in the relation. A graph is a geometric picture of the relation.

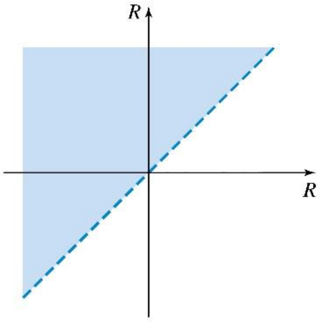
EXAMPLE 6. $A=\{1,2,3\}$, $B=\{1,2,3,4\}$

$$\begin{aligned} < &= \{(x,y) \mid x \in A,\ y \in B,\ x+m=y \text{ for some positive } m\} \\ &= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} \end{aligned}$$

Graph of < :

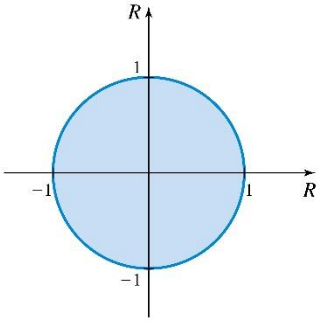


EXAMPLE 7. The following is a graph of < on ℝ:



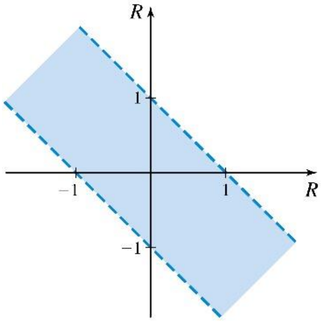
EXAMPLE 8. ℝ = real numbers

$$\rho = \{(x,y) \mid x^2 + y^2 \le 1\}$$



EXAMPLE 9. $\mathbb{R}$ = real numbers

$$\rho = \{(x,y) \mid |x+y| < 1\}$$



Definition 14. Given that ρ is a relation from A to B, the ***domain*** of ρ, D_ρ, and the ***range*** of ρ, R_ρ, are defined as follows:

a) $D_\rho = \{a \mid a \in A \text{ and } \exists y \in B, (a,y) \in \rho\}$

The domain is the set of all first coordinates of ordered pairs in ρ.

b) $R_\rho = \{b \mid b \in B \text{ and } \exists x \in A, (x,a) \in \rho\}$

The range is the set of all second coordinates of ordered pairs in ρ.

EXAMPLES.

a) In EXAMPLE 4: $D_\rho = \{1,2,3\}$, $R_\rho = \{1,4,9\}$

b) In EXAMPLE 5: $D_\rho = \mathbb{R}$, $R_\rho = \mathbb{R}$

This follows since every real number is less than some other real number, and every real number is greater than some real number.

Definition 15. Given that ρ is a relation from A to B, the ***inverse*** of ρ, denoted ρ^{-1}, is a relation from B to A defined:

$$\rho^{-1} = \{(b,a) \mid (a,b) \in \rho\}$$

Thus $(b,a) \in \rho^{-1} \leftrightarrow (a,b) \in \rho$. The relation ρ^{-1} results from interchanging the ordered pairs in ρ.

EXAMPLE 10. $A=\{1,2,3\}$, $B=\{1,2,3,4\}$

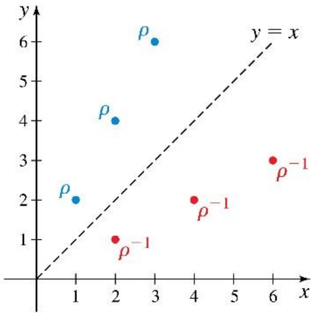
$$\begin{aligned} < &= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} \\ (<)^{-1} &= \{(2,1),(3,1),(4,1),(3,2),(4,2),(4,3)\} \end{aligned}$$

Perhaps you discovered that $(<)^{-1} = >$. (This is set equality.)

EXAMPLE 11. $A=\{1,2,3\}$, $B=\{1,2,3,4,5,6\}$

$$\rho = \{(x,y) | y = 2x\} = \{(1,2),(2,4),(3,6)\}$$

Note the graph of ρ with the blue dots:



Now

$$\begin{aligned} \rho^{-1} &= \{(y,x) | y = 2x\} \\ &= \{(2,1),(4,2),(6,3)\} \end{aligned}$$

Note the graph of ρ^{-1} with the red dots. We can also describe ρ^{-1} as follows:

$$\rho^{-1} = \{(x,y) | x = 2y\}$$

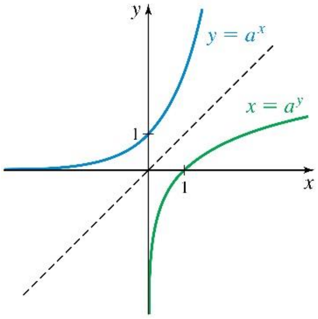
This is in keeping with the custom of having the first coordinate on the horizontal or *x*-axis. The graph of ρ^{-1} is thus the mirror image or reflection across the line *y* = *x* of the graph of ρ. To find a sentence to describe ρ^{-1}, we not only interchange the variables *y* and *x*, but we *rename* the variable *x* for *y* and *y* for *x*.

EXAMPLE 12. Given ℝ = real numbers and $\rho = \{(x,y) | y = \log_a x, \text{ or } x = a^y\}$:

a) Describe ρ^{-1}.

$$\rho^{-1} = \{(x, y) | x = \log_a y, \text{ or } y = a^x\}$$

b) Graph ρ and ρ^{-1} using the same set of axes.



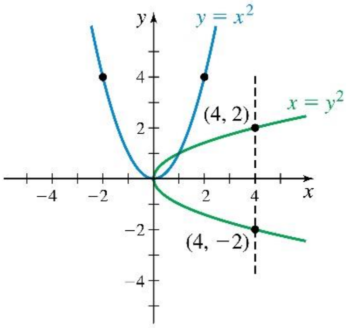
EXAMPLE 13. Given ℝ = real numbers and

$$\sigma = \{(x, y) | y = x^2\}.$$

a) Describe σ^{-1}:

$$\sigma^{-1} = \{(x, y) | x = y^2\}.$$

b) Graph σ and σ^{-1} using the same set of axes.



Exercise Set 3.3

Let $A = \{a, b, c\}$, $B = \{5, 6\}$. Find:

1. $A \times B$ 2. $B \times A$ 3. $A \times A$ 4. $B \times B$ 5. Does $A \times B = B \times A$?

Write three ordered pairs in each relation.

6. $L=\{(x,y)|x\in\mathbb{R}, y\in\mathbb{R}, x^2+y^2=1\}$

7. $M=\{(x,y)|x\in\mathbb{R}, y\in\mathbb{R}, x<y^2\}$

8. Find the domain and range of the relations in Examples 6-9.

9. a) Graph $\left\{(x,y)|x\in\mathbb{R}, y\in\mathbb{R}, \frac{x^2}{4}+\frac{y^2}{25}=1\right\}$.

   b) Find the domain and range.

Prove.

10. $(a,a)=\{\{a\}\}$

11. $(a,b)=(b,a)\leftrightarrow a=b$

12. Determine whether true or false.

    a) $(a,b)=(b,a)$ b) $\{a,b\}=\{b,a\}$

    c) $(a,b)=\{a,b\}$ d) $(a,a)=\{a\}$

13. Given $A=\{a,b\}$, $B=\{a,b,c\}$, $C=\{2,3\}$. Find:

    a) $A\times B$ b) $A\times C$ c) $B\times C$; compare $A\times C$ and $B\times C$

    d) $B\cup C$ e) $A\times(B\cup C)$ f) $(A\times B)\bigcup(A\times C)$; compare $A\times(B\cup C)$ and $(A\times B)\bigcup(A\times C)$

    g) $B\cap C$ h) $A\times(B\cap C)$ i) $(A\times B)\bigcap(A\times C)$; compare $A\times(B\cap C)$ and $(A\times B)\bigcap(A\times C)$

Prove: For any sets *A*, *B*, *C*,

14. $A\subseteq B\rightarrow A\times C\subseteq B\times C$

15. $A\times(B\cup C)=(A\times B)\cup(A\times C)$

16. $A\times(B\cap C)=(A\times B)\cap(A\times C)$

17. $A\times A=B\times B\rightarrow A=B$

18. Consider the following relation ρ from *A* to *B*:

$A=\{a,b\}$, $B=\{2,3,4\}$, $\rho=\{(a,2),(a,3),(b,4)\}$

Find:

a) D_ρ b) R_ρ c) ρ^{-1} d) $D_{\rho^{-1}}$ e) $R_{\rho^{-1}}$

f) Compare D_ρ with $R_{\rho^{-1}}$ g) Compare R_ρ with $D_{\rho^{-1}}$

19. Prove: If ρ is a relation from *A* to *B*, then $D_{\rho^{-1}}=R_\rho$ and $R_{\rho^{-1}}=D_\rho$.

Graph each of the following relations on ℝ.

20. $\{(x,y)|\,y=3x-1\}$

21. $\{(x,y)|\,|x|-|y|\le 1\}$

22. $\{(x,y)|\,x^2+y^2\le 4\}$

23. $\{(x,y)|\,x^2+y^2>4\}$

24. $\{(x,y)|\,|x|+|y|\le 2\}$

25. $\{(x,y)|\,y\ge x^2\}$

26. $\{(x,y)|\,x^2+y^2=4\}\cup\left\{(x,y)|-2\le x\le 2 \text{ and } y=-\sqrt{2-x^2}\right\}\cup\{(2,2),(-2,2)\}$

For each of the following relations ρ on ℝ describe ρ^{-1} and graph ρ and ρ^{-1} using the same set of axes.

27. $\rho=\{(x,y)|\,y=3x-1\}$

28. $\rho=\{(x,y)|\,y=x^3\}$

29. $\rho=\left\{(x,y)|\,\dfrac{x^2}{9}+\dfrac{y^2}{4}=1\right\}$

30. $\rho=\{(x,y)|\,y=\sin x\}$

31. $\rho=\{(x,y)|\,x^2+y^2=1\}$

32. $\rho=\{(x,y)|\,xy=1\}$

33. $\rho=\{(x,y)|\,y=\sin x \text{ and } 0\le x\le 2\pi\}$

34. $\rho=\{(x,y)|\,|x|+|y|=2\}$

3.4 EQUIVALENCE RELATIONS

In this section, we will be considering relations *on* a set *A*; that is, subsets of $A\times A$. Below are some relations. We will refer to these throughout this section.

Relation	**Set**
ρ_1 : "equal," $\{(1,1),(2,2),(3,3)\}$	$\{1,2,3\}$
ρ_2 : "on the same puzzle piece"	Set of atoms that make up puzzle
ρ_3 : $<$	$\mathbb{R}$
ρ_4 : $\perp$	Set of lines in a plane
ρ_5 : "is the father of"	People

Definition 16. Let ρ be a relation on *A*.

(R) ρ is ***reflexive*** iff for every $a\in A, (a,a)\in\rho$ (or, $a\rho a$). (Every element of *A* is related to itself.)

EXAMPLES.

1) ρ_1 on $\{1,2,3\}$ is reflexive, but $\sigma=\{(1,1),(2,2)\}$ on $\{1,2,3\}$ is *not* reflexive since $(3,3)\notin\sigma$. The universal quantifier is quite important here.

2) ρ_2 is reflexive since each atom belongs to the same puzzle piece as itself.

3) ρ_3 is not reflexive since, for example, $2\not<2$.

4) ρ_4 is not reflexive. A line cannot be perpendicular to itself.

5) ρ_5 is *not* reflexive. One cannot be the father of himself.

Definition 17.

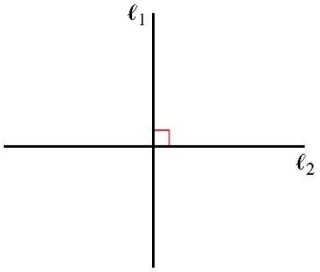
(S) ρ is ***symmetric*** iff for every $a,b\in A, (a,b)\in\rho\rightarrow(b,a)\in\rho$. (For any $a,b\in A$, if *a* is related to *b*, then *b* is related to *a*.)

EXAMPLES.

1) $=$is symmetric. For any $a, b \in A, a = b \to b = a.$ The relation $\{(1,2),(3,3),(2,1),(2,3)\}$ is not symmetric.

2) ρ_2 is symmetric. If *a* is on the same puzzle piece as *b*, then *b* is on the same piece as *a*.

3) < is *not* symmetric. $2 < 3$ and $3 \not< 2$.



4) $\perp$ is symmetric. If $l_1 \perp l_2,$ then $l_2 \perp l_1.$

5) ρ_5 is *not* symmetric. *a* is the father of *b* $\not\to$ *b* is the father of *a*.

Definition 18.

(T) ρ is ***transitive*** iff for any $a, b, c \in A, (a,b) \in \rho$ and $(b,c) \in \rho \to (a,c) \in \rho.$ (For any $a,b,c \in A,$ if *a* is related to *b* and *b* is related to *c*, then *a* is related to *c*.)

We might think of the transitive property as the "bridge" property.

EXAMPLES.

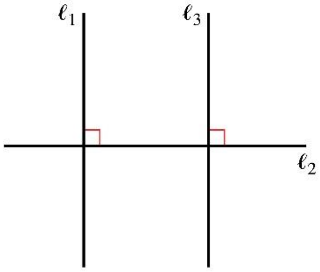
1) = is transitive. If *a* = *b* and *b* = *c*, then *a* = *c*. The relation $\{(2,3),(3,1),(2,1)\}$ is also transitive but $\{(1,2),(2,3)\}$ is not.

2) ρ_2 is transitive. If *a* is on the same piece as *b* and *b* is on the same piece as *c*, then *a* is on the same piece as *c*.

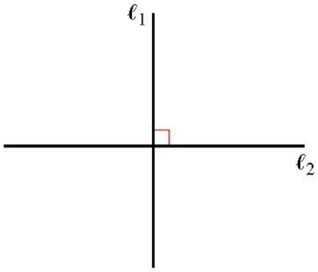
3) < is transitive. If *a* < *b* and *b* < *c*, then *a* < *c*.

4) $\perp$ is not transitive. We illustrate this in two ways.

a) $l_1 \perp l_2$ and $l_2 \perp l_3$ and $l_1 \not\perp l_3$



b) $l_1 \perp l_2$ and $l_2 \perp l_1$ and $l_1 \not\perp l_1$



Part b) illustrates an important point about quantifiers. When we say, "For any $a, b, c, \ldots$," we *do not* exclude the possibility that they may be equal.

5) ρ_5 is not transitive. For *a* is the father of *b* and *b* is the father of *c* $\nrightarrow$ *a* is the father of *c*. That is, a grandfather is not the father of his grandson.

Definition 19. A relation ρ on *A* is an ***equivalence relation*** iff it is reflexive, symmetric, and transitive.

Accordingly, an equivalence relation is also called an ***R-S-T relation***.

EXAMPLES. The relations = and ρ_2 are equivalence relations. The relations ρ_3 to ρ_5 are *not*.

Equivalence classes. When we have an equivalence relation $\rho,$ it is typical to read "$a\rho x,$" or "$(a,x) \in \rho$" as "*a* is equivalent to *x*."

Definition 20. Let ρ be an equivalence relation on *A*. For any $a \in A,$ the ***equivalence class of a***, $E(a),$ is defined:

$$E(a) = \{x | x \in A \text{ and } (a,x) \in \rho, \text{ or } a\rho x\}$$

(The equivalence class of *a*, $E(a),$ is the set of all elements equivalent to *a*.)

Notations such as $cl(a)$ and [*a*] are also used for equivalence classes. The following examples refer to the relations described in previous exercises or examples.

EXAMPLES. List or describe the equivalence classes.

	Set	Equivalence Relation	Equivalence Classes
1)	$\{1,2,3,4\}$	$\rho_6 = \{(1,1),(2,2),(3,3),(4,4),(4,2),(2,4)\}$	$E(1)=\{1\}$ $E(2)=\{2,4\}=E(4)$ $E(3)=\{3\}$
2)	Atoms of a puzzle ρ_2	"on the same puzzle piece"	The pieces of the puzzle.
3)	Triangles in a plane ρ_{11}	"similarity"	Each equivalence class is the set of triangles similar to a given one; that is, whose sides are proportional.

Now look carefully at each collection of equivalence classes we have considered. See how many conjectures you can make. Then consider the following theorem.

Theorem 16. Let *A* be a nonempty (universal) set and ρ be an equivalence relation on *A*. Then:

a) For any $a \in A, a \in E(a)$

(For each equivalence class $E(a)$, $E(a) \neq \varnothing$.)

b) $E(a) = E(b) \leftrightarrow (a,b) \in \rho$

(Equivalence classes *E*(*a*) and *E*(*b*) are equal iff *a* is equivalent to *b*.)

c) For any *E*(*a*) and *E*(*b*), $E(a) = E(b)$ or $E(a) \cap E(b) = \varnothing$

(Equivalence classes are either equal or disjoint.)

Proof of a): Let $a \in A$. Then $(a,a) \in \rho$ by the reflexive property. Thus, by Definition 20, $a \in E(a)$.

Proof of b):

1) Prove: $E(a)=E(b)\rightarrow(a,b)\in\rho$. Assume $E(a) = E(b)$. By part a) we know $a\in E(a)$. Since $E(a) = E(b)$, $a\in E(b)$. Then, by Definition 20, $(b,a)\in\rho$ and by symmetry $(a,b)\in\rho$.

2) Prove: $(a,b)\in\rho\rightarrow E(a)=E(b)$. Left as an exercise.

Proof of c): Let $E(a)$ and $E(b)$ be arbitrary equivalence classes. We know by the tautology $P\vee\sim P$ that

$$E(a) = E(b) \text{ or } E(a) \neq E(b).$$

Thus, all we really need to prove is that

$$E(a)\neq E(b)\rightarrow E(a)\cap E(b)=\varnothing.$$

We prove this by proving its contrapositive:

$$E(a)\cap E(b)\neq\varnothing\rightarrow E(a)=E(b).$$

Assume $E(a)\cap E(b)\neq\varnothing$. Then there exists an x such that $x\in E(a)$ and $x\in E(b)$. Thus, by Definition 20,

$$(a,x)\in\rho \text{ and } (b,x)\in\rho.$$

Then by symmetry $(x,b)\in\rho$ and by transitivity, $(a,b)\in\rho$. So, by part b), $E(a) = E(b)$. ■

Exercise Set 3.4

Translate the definition of each of the following to logical symbolism and write the negation.

1. reflexive
2. symmetric
3. Transitive

Consider:

	Relation	Set
ρ_6 :	$\{(1,1),(2,2),(3,3),(4,4),(4,2),(2,4)\}$	$\{1,2,3,4\}$
ρ_7 :	$\{(2,2),(3,3),(4,2),(2,4)\}$	$\{1,2,3,4\}$
ρ_8 :	$\{(4,3),(3,4),(3,3)\}$	$\{1,2,3,4\}$
ρ_9 :	$\leq$	$\mathbb{R}$
ρ_{10} :	$\cong$ (Congruence)	Set of triangles in a plane
ρ_{11} :	$\sim$ (Similar)	Set of triangles in a plane
ρ_{12} :	"has the same area as"	Set of triangles in a plane
ρ_{13} :	"is a brother of"	Set of all living people
ρ_{14} :	"is a brother of" (Assume that a man is a brother of himself)	Set of all living men

4. Which of the above are reflexive?

5. Which of the above are symmetric?

6. Which of the above are transitive?

7. Which of the above are equivalence relations?

Each exercise refers to an equivalence relation in the previous examples or exercises. List or describe the equivalence classes.

8. ρ_1
9. ρ_{10}
10. ρ_{12}
11. ρ_{14}

12. In part b) of Theorem 16, prove: $(a,b) \in \rho \to E(a) = E(b)$. *Hint:* Remember that you are proving set equality.

3.5 PARTITIONS

Reread Theorem 16 in Section 3.4. We can now show that equivalence classes satisfy the following definition.

Definition 21. A ***partition*** of a nonempty set (universal set) *U* is a collection $\{A_i\}_{i\in F}$ of subsets of *U* such that:

1) For any A_i, $A_i \neq \varnothing$

2) For any A_i, A_j; $A_i = A_j$ or $A_i \cap A_j = \varnothing$

3) $\bigcup\limits_{i\in F} A_i = U$

That is, a partition is a collection of nonempty subsets which are disjoint and whose union is the whole set. The following are examples.

EXAMPLE 1. Let *A* = set of atoms of a puzzle. The pieces form a partition.

EXAMPLE 2. Let $\mathbb{R}$ = real numbers, $\mathbb{Q}$ = rationals, and *J* = irrationals. Then $\{\mathbb{Q}, J\}$ is a partition of $\mathbb{R}$.

EXAMPLE 3. Let $\mathbb{R}$ = real numbers, $\mathbb{Z}$ = integers, and

$$A_i = \{x \mid i \le x < x+1\}.$$

Then $\{A_i\}_{i\in\mathbb{Z}}$ is a partition of $\mathbb{R}$.

EXAMPLE 4. Let $U = \{1,2,3,4,5\}$.

a) $B_1 = \{1,2\}$, $B_2 = \{2,3,4\}$, $B_3 = \{5\}$

$\{B_i\}_{i=1}^{3}$ is not a partition because $B_1 \cap B_2 \neq \varnothing$.

b) $B_1 = \{1\}$, $B_2 = \{2,3\}$, $B_3 = \{5\}$

$\{B_i\}_{i=1}^{3}$ is not a partition because $\bigcup_{i=1}^{3} B_i \neq U$.

Theorem 17. Let ρ be an equivalence relation on a nonempty set *A*. Then $\{E(a)\}_{a\in A}$, the collection of all equivalence classes, is a partition of *A* (called the ***induced partition***).

Prove: We have to prove the following:

1) For any $a \in A, E(a) \neq \varnothing.$

2) For any $E(a), E(b), E(a) = E(b)$ or $E(a) \cap E(b) = \varnothing.$

3) $\bigcup_{i \in F} A_i = A.$

Proof. Parts 1) and 2) follow from Theorem 16. Part 3) is left as an exercise. ■

Conversely, we can take a partition and obtain an equivalence relation whose equivalence classes are the members of the partition.

Theorem 18. Suppose $\{A_i\}_{i \in P}$ is a partition of *A*. We define a relation as follows:

$$(a,b) \in \rho \leftrightarrow a, b \in A_i \text{ for some } i \in F. \qquad (1)$$

Then ρ is an equivalence relation (called the ***induced relation***).

Proof. (Reflexive): Let $a \in A.$ Since $\bigcup_{i \in F} A_i = A$ it follows that $a \in A_i$ for some $i \in F.$ Thus $(a,a) \in \rho$ by (1). The symmetric and transitive properties are left as exercises. ■

Thus, partitions and equivalence classes go hand in hand. Every equivalence relation induces a partition. Every partition induces an equivalence relation. When you think of one, you should think of the other.

EXAMPLE. Find the induced equivalence relation and the set on which it is defined for the partition

$$\{\{a,b\},\{c\},\{d\}\}$$

From $\{a,b\}$ we get $(a,a),(b,b),(a,b),(b,a) \in \rho$

From $\{c\}$ we get $(c,c) \in \rho$

From $\{d\}$ we get $(d,d) \in \rho$

Thus, the relation is $\rho = \{(a,a),(b,b),(a,b),(b,a),(c,c),(d,d)\}$ and is defined on

$\{a,b,c,d\}.$

Why Do We Study Equivalence Relations? There are several answers to this question. First, if you think of all the equivalence relations we have already considered, you get some idea of how much this notion pervades mathematics. Second, future study in mathematics will necessitate the use of equivalence relations. Stated as simply as possible, a set may lack a "certain property." We form an equivalence relation on the set of ordered pairs and the result will have the missing property. Third, the notion of equivalence relation pervades everyday life much more than you realize. For example, equivalence classes (partitions) are a way of categorizing. Think of students in your class and the partitions resulting from the following equivalence relations:

"in the same row as"

"gets the same grade as"

"is the same sex as"

"is the same height as"

Another example occurs in the educational process. When you learn a "concept," you form partitions or equivalence classes. In learning "color," you mentally separate objects into "all the red ones," "all the green ones," and so on.

We close this section by considering an important type of equivalence relation on the set of integers, $\mathbb{Z}$. Let

$$\rho = \left\{(x,y) \middle| x \in \mathbb{Z}, y \in \mathbb{Z}, x - y = 3k, \text{ for some } k \in \mathbb{Z} \text{ (or } x - y \text{ is divisible by 3)}\right\}.$$

The equivalence classes are as follows:

$$E(0) = \{\ldots,-9,-6,-3,0,3,6,9,\ldots\} = E(-3) = E(9) = E(3k), \text{ for any } k \in \mathbb{Z}.$$

$$E(1) = \{\ldots,-8,-5,-2,1,4,7,10,\ldots\} = E(-5) = E(7) = E(3k+1), \text{ for any } k.$$

$$E(2) = \{\ldots,-7,-4,-1,2,5,8,11,\ldots\} = E(-7) = E(8) = E(3k+2), \text{ for any } k.$$

The partition, or set of equivalence classes, has just three members, $\{E(0), E(1), E(2)\}$, sometimes referred to as $\mathbb{Z}(\text{mod}\,3)$ or "integers modulo 3."

Exercise Set 3.5

1. Prove that $\mathbb{Z}(\text{mod}\,3)$ is an equivalence relation.

2. Consider the relation on $\mathbb{Z}$ defined as follows:

$$\rho = \{(x,y) | x \in \mathbb{Z}, y \in \mathbb{Z}, x - y = 2k \text{ for some } k \in \mathbb{Z}\}$$

a) Prove ρ is an equivalence relation.

b) Find $E(0)$ and $E(1)$. Give them other names.

The set of equivalence classes $\{E(0), E(1)\}$ is called $\mathbb{Z}(\text{mod}\,2)$.

3. Consider the relation ρ on $\mathbb{Z}$ defined as follows:

$$\rho = \{(x,y) | x \in \mathbb{Z}, y \in \mathbb{Z}, x - y = 4k, \text{ for some } k \in \mathbb{Z}\}$$

a) Prove ρ is an equivalence relation.

b) Find $E(0), E(1), E(2), E(3)$. Give other names for these equivalence classes.

The set $\{E(0), E(1), E(2), E(3)\}$ is called $\mathbb{Z}(\text{mod}\,4)$.

4. Generalize the previous work to $\mathbb{Z}(\text{mod}\,n)$. For fixed $n \in \mathbb{N}$, define

$$\rho = \{(x,y) | x \in \mathbb{Z}, y \in \mathbb{Z}, x - y = nk, \text{ for some } k \in \mathbb{Z}\}$$

a) Prove ρ is an equivalence relation.

b) Describe $E(0), E(1), \ldots, E(n-1)$.

c) How many equivalence classes are there?

5. Given a set *A*. Prove:

a) $A \times A$ is an equivalence relation on *A*.

b) $i_A = \{(a,a) | a \in A\}$ is an equivalence relation on *A*. The relation i_A is called the ***identity relation*** on *A*.

6. Find relations different from any previously discussed which are:

a) R, S, T	b) R, S, not T	c) R, not S, T
d) not R, S, T	e) R, not S, not T	f) not R, S, not T
g) not R, not S, T	h) not R, not S, not T	

7. Find the flaw in the following "proof."

   Let ρ be a relation on A.

   THEOREM: ρ is symmetric and ρ is transitive $\rightarrow \rho$ is reflexive.

   Proof. If $(a,b) \in \rho$, then $(b,a) \in \rho$ by symmetry.

   Then $(a,b) \in \rho$ and $(b,a) \in \rho \rightarrow (a,a) \in \rho$ by transitivity. Thus $(a,a) \in \rho$, so ρ is reflexive. ■

8. Decide which of the following collections $\mathcal{P}$ is a partition on the given set A. If $\mathcal{P}$ is a partition, describe the induced equivalence relation.

   a) $A = \{1,2,3,4\}$; $\mathcal{P} = \{\{1,2\},\{3\},\{4,1\}\}$

   b) $A = \{1,2,3,4\}$; $\mathcal{P} = \{\{1,2,3\},\{4\},\{5\}\}$

   c) $A = \{\text{a, b, c, d, e}\}$; $\mathcal{P} = \{\{\text{a, b, c}\},\{\text{d, e}\}\}$

9. Suppose ρ_1 and ρ_2 are equivalence relations on a set A. Prove or disprove:

   a) $\rho_1 \cup \rho_2$ is an equivalence relation.

   b) $\rho_1 \cap \rho_2$ is an equivalence relation.

10. Let ρ be a relation on A. Prove:

    a) ρ is reflexive $\rightarrow \rho^{-1}$ is reflexive

    b) ρ is symmetric $\rightarrow \rho^{-1}$ is symmetric

    c) ρ is transitive $\rightarrow \rho^{-1}$ is transitive

d) ρ is an equivalence relation $\to \rho^{-1}$ is an equivalence relation

e) ρ is symmetric $\leftrightarrow \rho^{-1} \subseteq \rho$

11. Prove Part 3) of Theorem 17.

12. Prove the symmetric and transitive properties of Theorem 18.

3.6 FUNCTIONS

You have probably encountered functions in previous study. We can use our set theory to give a precise definition of a function as a special kind of relation. This may be new to you. Before we give this definition, let us motivate it by considering intuitive ways you may have studied functions.

A Function As a Rule. A *function f* from a set *A* into a set *B* is a rule which assigns to each element of a set *A*, called the domain, a unique element in a set *B*. Often, we have a formula like

$$f(x) = 3^x + \frac{1}{x}$$

which describes the rule. We can think of it with "blank spaces"

$$f(\square) = 3^\square + \frac{1}{\square}.$$

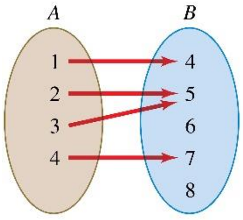
We then have a recipe. Given an element, say *a*, in *A*, we "plug" it in the blanks to find the element it is assigned to. Usually the domain is the set of all numbers that can be "plugged in." For this function, the domain is $\{x | x \neq 0\}$.

We can also get the "function machine" idea out of this. For each "input" we get exactly one "output."

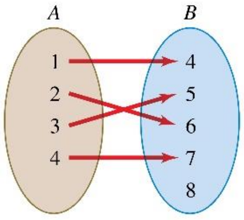
The notion of a function as a rule is encountered most in physics, chemistry, and related sciences.

Function As a Correspondence. A function from a set *A* into a set *B* is a correspondence which assigns to each element in a set *A* a unique element in a set *B*.

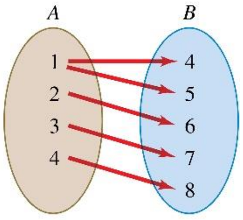
EXAMPLES.



A function
(1)



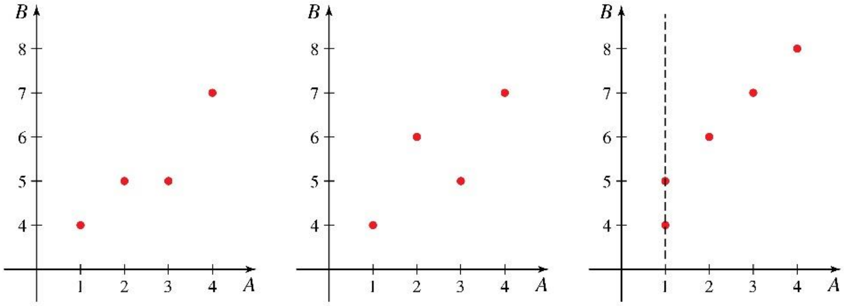
A function
(2)



Not a function
(3)

Note that in (1) 2 is assigned to 5 and no other object in *B*; 3 is assigned to 5, and no other object in *B*. But in (3) 1 is assigned to both 4 and 5 so (3) is not a function.

A Function As a Set of Ordered Pairs. Consider these graphs of the previous correspondences.



A function (1)

A function (2)

Not a function (3)

Recall that the sets of ordered pairs pictured in (1) and (2) are functions. Note that for (1) and (2) no vertical line crosses the graph more than once. But for (3) there is a vertical line which crosses the graph more than once. Hence it is not a function. Thus, we could define a ***function* as a set of ordered pairs for which no two distinct ordered pairs have the same first coordinate**.

A Function As a Relation. We can now give a precise set-theoretic definition of function.

Definition 22. Let *A* and *B* be sets. A ***function f from A to B***, denoted $f : A \to B,$ is a relation *f* from *A* to *B* such that

a) $D_f = A$

b) For every $a \in A, b_1, b_2 \in B, (a,b_1) \in f$ and $(a,b_2) \in f \to b_1 = b_2$.

We also say that $(x, y) \in f$ means $f(x) = y$ and $f(x)$(read "f of x") is called **the *value* of *f* at *x***. We then reformulate part b) as

b$'$) For every $a_1, a_2 \in A, a_1 = a_2 \to f(a_1) = f(a_2)$.

Part b) or b$'$) insures uniqueness. To see this, we might write the contrapositive

b$''$) For every $a_1, a_2 \in A,\ f(a_1) \neq f(a_2) \to a_1 \neq a_2$.

In truth, there is much controversy among mathematicians about the definition of function, as will be shown in the examples and references in the exercises. One way to evidence this is to go to the library, pick five mathematics texts at random, and compare their definitions of *function*.

The function $f:\mathbb{R}\to\mathbb{R}$

$$f=\{(x,y)\mid y=3^x+\sin x\}$$

could also be represented as follows:

$$f=\{(x,f(x))\mid f(x)=3^x+\sin x\},\text{ or}$$

$$f:\mathbb{R}\to\mathbb{R}\text{ described by } f(x)=3^x+\sin x.$$

Note that under our precise definition of function the following terminology is incorrect (even though often used):

“the function $y=3^x+\sin x$” (i)

“the function $f(x)=3^x+\sin x$” (ii)

“the function $f(x)$” (iii)

This is because (i) and (ii) refer to sentences, not sets, and (iii) refers to the value of *f* at *x*, it is not a set of ordered pairs.

EXAMPLES. Find the domain and range and decide which are functions. Let

$A=\{a,b,c\}$, $B=\{b,c,d,e\}$ for 1) and 2).

1) $f:A\to B$ by $f=\{(a,b),(a,c),(b,d),(c,e)\}$

$$D_f=\{a,b,c\},R_f=\{b,c,d,e\}.$$

The notation $f:A\to B$ is false since $(a,b)\in f$ and $(a,c)\in f$; *f* is not a function.

2) $g:A\to B$ by $g=\{(a,c),(b,d),(c,e)\}$

$$D_g=\{a,b,c\},R_g=\{c,d,e\}.$$

The notation $g:A\to B$ is correct; *g* is a function. Note that $R_g \neq B$; this does not prevent it from being a function.

3) $h:\mathbb{R}\to\mathbb{R}$ by $h(x)=\dfrac{1}{x^2}$

$D_h=\{x|x\neq 0\}=\mathbb{R}-\{0\}, R_h=\{y|y>0\}.$

This is not a function since $\mathbb{R}-\{0\}\neq\mathbb{R}.$ But, the following is a function:

$$h_1:(\mathbb{R}-\{0\})\to\mathbb{R} \text{ by } h_1(x)=\frac{1}{x^2}.$$

Exercise Set 3.6

Graph each of the following relations on $\mathbb{R}$. Then decide which are functions.

1. $\{(x,y)|y=x^2\}$
2. $\{(x,y)|x=y^2\}$
3. $\{(x,y)|y=4\}$
4. $\{(x,y)|x=-2\}$
5. $\{(x,y)|y>x\}$
6. $\{(x,y)|4x^2+9y^2=36\}$

Find the domain and range and decide which are functions. Let $A=\{\text{a, b, c, d}\}$ and $B=\{1,2,3,4,5\}$ for Exercises 7-11.

7. $f_7:A\to B$ by $f_7=\{(\text{a}, 1),(\text{b}, 2),(\text{c}, 3)\}$
8. $f_8:A\to B$ by $f_8=\{(\text{a}, 1),(\text{b}, 2),(\text{c}, 3),(\text{d}, 4)\}$
9. $f_9:A\to A$ by $f_9=\{(\text{a}, \text{a}),(\text{b}, \text{b}),(\text{c}, \text{c}),(\text{d}, \text{d})\}$
10. $f_{10}:B\to A$ by $f_{10}=\{(1, \text{a}),(2, \text{b}),(3, \text{c}),(4, \text{c}),(5, \text{d})\}$
11. $f_{11}:A\to B$ by $f_{11}=\{(\text{a}, 1),(\text{b}, 1),(\text{c}, 3),(\text{d}, 5)\}$

Definition 23. A function $f:A\to B$ is ***onto***, ***surjective***, or a ***surjection*** iff for every element *b* in *B* there exists an element *a* in *A*, such that $f(a)=b$.

12. a) Translate the definition of an onto function to logical symbolism.

    b) Write a negation of the definition of an onto function.

13. Which of the functions in Exercises 1 and 3 are from the set of real numbers onto the set of real numbers?

14. Which of the functions in Exercises 7-11 are onto set *B*?

Definition 24. A function $f : A \to B$ is ***one-to-one, injective,*** or an ***injection*** iff for every element *a* and every element *b* of set *A*, if $f\left(a\right) = f\left(b\right),$ then $a = b$.

15. a) Translate the definition of a one-to-one function to logical symbolism.

    b) Write a negation of the definition of a one-to-one function.

16. Which of the functions in Exercises 1 and 3 are one-to-one?

17. Which of the functions Exercises 7-11 are one-to-one?

18. Let *A* be the finite set $\{a_1, a_2, \ldots, a_n\}$. Use mathematical induction to prove that the set of all functions from set *A* into set *A* which are both one-to-one and onto has $n!$ elements in it.

APPENDIX

The Real Number System

ℝ denotes the real number system. ℝ is equipped with an algebraic structure, some properties of which are listed below. When a property also holds for the set of natural numbers ℕ, the set of integers ℤ, or the set of rationals ℚ, we list the appropriate symbol to the right.

Addition.

A1) (Closure) $\forall a \forall b \left(a+b \in \mathbb{R}\right)$ ℕ, ℤ, ℚ

A2) (Associative Law) $\forall a \forall b \forall c \left[a+\left(b+c\right)=\left(a+b\right)+c\right]$ ℕ, ℤ, ℚ

A3) (Additive Identity) There exists an element denoted 0, such that

for every a, $a + 0 = 0 + a = a$. ℤ, ℚ

A4) (Additive Inverse) $\forall a \exists\left(-a\right)\left[a+\left(-a\right)=\left(-a\right)+a=0\right]$ ℤ, ℚ

A5) (Commutative Law) $\forall a \forall b \left(a+b=b+a\right)$ ℕ, ℤ, ℚ

Multiplication.

M1) (Closure) $\forall a \forall b \left(ab \in \mathbb{R}\right)$ ℕ, ℤ, ℚ

M2) (Associative Law) $\forall a \forall b \forall c \left[a\left(bc\right)=\left(ab\right)c\right]$ ℕ, ℤ, ℚ

M3) (Multiplicative Identity) There exists an element, denoted 1,

such that $\forall a \left(a \cdot 1 = 1 \cdot a = a\right)$. ℕ, ℤ, ℚ

M4) (Multiplicative Inverse) $\forall a \left[a \neq 0 \rightarrow \exists a^{-1}\left(a \cdot a^{-1} = a^{-1} \cdot a = 1\right)\right]$ ℚ

M5) (Commutative Law) $\forall a \forall b \left(ab=ba\right)$ ℕ, ℤ, ℚ

M6) (Distributive Law) $\forall a \forall b \forall c \left[a\left(b+c\right)=ab+ac\right]$ ℕ, ℤ, ℚ

M7) $1 \neq 0$

Order Properties

O1) (Trichotomy Law) For any two real numbers a, b exactly one of the following is true:

a) $a < b$, b) $a = b$, c) $a > b$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

For example, $a \neq 0 \rightarrow a < 0 \vee a > 0$.

O2) (Transitive Law) $\forall a \forall b \forall c \left[\left(a < b \wedge b < c \right) \rightarrow a < c \right]$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

O3) $\forall a \forall b \forall c \left(a < b \rightarrow a + c < b + c \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

$\forall a \forall b \forall c \left(a \leq b \rightarrow a + c \leq b + c \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

O4) $\forall a \forall b \forall c \left(a < b \wedge c > 0 \rightarrow ac < bc \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

$\forall a \forall b \forall c \left(a \leq b \wedge c > 0 \rightarrow ac \leq bc \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

O5) $\forall a \forall b \forall c \left(a < b \wedge c < 0 \rightarrow ac > bc \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

$\forall a \forall b \forall c \left(a \leq b \wedge c < 0 \rightarrow ac \geq bc \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

O6) $1 > 0$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Other Properties

P1) $\forall a \forall b \left[a < b \leftrightarrow \exists p > 0 \left(a + p = b \right) \right]$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

P2) **Definition.**

$\forall a \left[\left(a \geq 0 \rightarrow |a| = a \right) \wedge \left(a < 0 \rightarrow |a| = -a \right) \right]$, then

$\forall a \left(a \neq 0 \rightarrow |a| > 0 \right)$ $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

P3) $\forall a \left(a \cdot 0 = 0 \cdot a = 0 \right)$ $\mathbb{Z}, \mathbb{Q}$

ANSWERS TO SELECTED EXERCISES

Exercise Set 1.1

1. $\{1,2,3,4,5,6,7,8\}$ **3.** $\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$ **5.** $\{x \mid -1 \le x \le 2,\ x \text{ an integer}\}$

6. $\{x \mid x > 9,\ x \text{ an integer}\}$ or $\{x \mid x \ge 10,\ x \text{ an integer}\}$

7. $\{x \mid x = 10 \cdot k \text{ for some natural number } k\}$ **9.** $\notin$ **11.** $\in$ **13.** $\subseteq$ **15.** $=$ **16.** $B = H, E = F,$

$D = G$ **17.** True **19.** False **21.** True **23.** $\varnothing$ **25.** $\varnothing$ **27.** $\{x \mid x < -1\}$ **29.** $\{0,1,2,3,4,\ldots\}$
31. A

Exercise Set 1.2

1a. A **1b.** $\{0,8\}$ **1c.** $\{0,2,3,8\}$ **1d.** $\{0,2,3,8,10\}$ **1e.** $\{0,8\}$ **1f.** $\{0,2,3,8\}$ **1g.** $\{1\}$

1h. $\{1\}$ **1i.** U **1j.** U **2.** $\mathbb{Q}$ **3.** $\varnothing$ **4.** $\varnothing$ **5.** $\varnothing$ **7.** J **8.** $\mathbb{Z}$ **9.** $\mathbb{Q}$

10. $\{x \mid x = 2k+1, \text{ for some integer } k\}$, or $\{\ldots,-5,-3,-1,1,3,5,7,\ldots\}$ **11a.** $\varnothing$ **11b.** $\mathbb{Z}$

12a. $\{\ldots,-9,-6,-3,0,3,6,9,\ldots\}$ **12b.** $\{\ldots,-8,-5,-2,1,4,7,10,\ldots\}$ **12d.** $\varnothing$ **12e.** $\varnothing$

12f. $\varnothing$ **12g.** $\mathbb{Z}$ **13.** $\{-1,-2\}$ **14.** $\varnothing$ **15.** $\varnothing$ **17.** $\{\ldots,-2,0,2,\ldots\}$ **18.** $\{2\}$ **19a.** $[2,3)$

19b. $\mathbb{R}$ **19c.** $[-1,4)$ **19d.** $[1,2)$ **19e.** $\{3\}$ **19f.** $\varnothing$ **19g.** $[-n,n]$ **19h.** $[-(n+1), n+1]$

20b. $\{\varnothing,\{0\}\}$ **20d.** $\{\varnothing\}$

Exercise Set 1.3

1. b, d, e, f, and g **2a.** x **2b.** n **2c.** x, y **2d.** x **2e.** x, y **3.** a, c **4.** $\{0,1,2,3\}$ **5.** $\{0,1\}$ **6.** $\varnothing$

7. $\{1,-2\}$ **8.** $\varnothing$ **9.** $\{-1\}$ **10.** $\varnothing$ **11.** $\{-1\}$ **12.** $\{-1\}$ **13.** $\{-1,-\tfrac{1}{2}\}$ **14.** $\{-1,-\tfrac{1}{2}\}$

15. $\varnothing$ **16.** $\{x \mid x \in \mathbb{R} \text{ and } x \ne -2\}$ **17.** False, $1^2 \ne 0$ **18.** False **19.** True **20.** True **21.** False,

$2^2 \ne 2$ **22.** True **23.** True **24.** True **25.** True **26.** True **27.** True **28.** True **29.** True

30. False, see Exercise 26 **31.** False **32.** False, $2+3 \ne 0$ **33.** True **34.** True

Exercise Set 1.4

1. F **2.** T **3.** T **4.** F **5.** F **6.** T **7.** $\sim P$; $\sim\left(2=3\right)$; $2 \neq 3$; It is false that 2 = 3; It is not true that 2 = 3 **8.** $\sim P$; *e* is not irrational; *e* is rational; It is false that *e* is irrational; It is not true that *e* is irrational **9.** T **10.** F **11.** $x \geq y$ **12.** $x \leq y$ **13.** $3 > y$ **14.** $z^2 < 1 + x$ **15.** T **16.** T **17.** F **18.** T

In 19 – 33 $P \to Q$ can replace “If *P*, then *Q*,” and vice versa.

19. If *n* is prime, then *n* has no factorization. **20.** If $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$. **21.** If $|r| < 1$, then $\lim_{n\to\infty}\left(a + ar + \cdots + ar^n\right) = a/\left(1-r\right)$. **22.** If $x \in \mathbb{N}$, then $x \in \mathbb{Z}$. **23.** If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges. **25.** $a \in \mathbb{Q} \to a \in \mathbb{R}$ **26.** $x \in \mathbb{Z} \to x \in \mathbb{Q}$ **28.** ℓ_1 parallel to $\ell_2 \to \ell_1 \cap \ell_2 = \varnothing$ **29.** *x* is a square $\to$ *x* is a rectangle. **30.** *x* is a triangle $\to$ *x* is a polygon. **31.** $x = y \to 3x = 3y$ **32.** $f\left(x\right) = x^2 \to f'\left(x\right) = 2x$ **33.** *x* is a square $\to$ *x* is not a triangle.

Exercise Set 1.5

1. F **2.** T **3.** F **4.** T **5.** T **6.** $x = 5 \leftrightarrow 2x = 10$ **7.** $x \in \mathbb{Q} \leftrightarrow \left(x = p/q \text{ where } p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0\right)$ **8.** $\{x_n\}_{n=1}^{\infty}$ has a limit $\leftrightarrow |x_m - x_n| \to 0$ as *m*, *n* go to infinity. **9.** $ab = 0 \leftrightarrow \left(a = 0 \vee b = 0\right)$ **10.** A triangle is isosceles $\leftrightarrow$ two sides are equal. If we used a variable *x* for “triangle” we could translate the sentence as: *x* is isosceles $\leftrightarrow$ *x* has two sides equal. **11.** $2x - 1 = 0 \leftrightarrow x = ½$ **12.** *f* is continuous $\leftrightarrow$ *f* is differentiable.

13. $\left(p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0\right) \to p/q \in \mathbb{Q}$ **14.** (*ABC* is a triangle $\wedge$ *ABC* is isosceles) $\to$ *ABC* has two equal sides. **15.** $\left(a, b, c, \text{ and } x \text{ are real numbers } \wedge a \neq 0 \wedge ax^2 + bx + c = 0 \wedge b^2 - 4ac = 0\right) \to$ the roots of $ax^2 + bx + c = 0$ are real and equal. **16.** $\sum_{n=1}^{\infty} u_n$ is convergent $\to \lim_{n\to\infty} u_n = 0$.

17. $\lim_{n\to\infty} u_n \neq 0 \to \sum_{n=1}^{\infty} u_n$ is not convergent. **18.** $a \in \mathbb{Z} \to$ (*a* is even $\vee$ *a* is odd). **19.** (*f* is differentiable $\wedge$ *g* is differentiable) $\to g \circ f$ is differentiable. **20.** (*u* and *v* are differentiable functions of *x*) $\to \left[uv \text{ is also differentiable } \wedge d\left(uv\right)/dx = u\left(dv/dx\right) + v\left(du/dx\right)\right]$.

21. $\left(x \in J \vee x \in \mathbb{Q}\right) \leftrightarrow x \in \mathbb{R}$ **22.** $\left(x \in {}^{-}\mathbb{N} \vee x \in \mathbb{N}_0\right) \leftrightarrow x \in \mathbb{Z}$

23. $x \in A \cap B \leftrightarrow (x \in A \wedge x \in B)$ **24.** $x \in A \cup B \leftrightarrow (x \in A \vee x \in B)$ **25.** $x \in A' \leftrightarrow x \notin A$

Exercise Set 1.6

1. $\forall x$, x is a triangle $\rightarrow x$ is a polygon. **2.** $\forall x$, x is a natural number $\rightarrow x$ is an integer.

3. $\forall x[x$ is a natural number $\rightarrow (x$ is even $\vee\, x$ is odd$)]$. **4.** $\exists x$, x is prime $\wedge\, x$ is even.

5. $\exists x, x = \lim\limits_{n \to \infty}(1/n)$. **6.** $\exists X\left[n \le X \le 2 \wedge \int_n^2 f(x)dx = (2-n)f(X)\right]$ **7.** $\exists p \exists q$, $p \cdot q = 32$

8. $\forall x \exists y$, $x < y$ **9.** $\exists y \forall x$, $x + 0 = y$ **10.** $\exists x \exists y$, x^y is irrational. **11.** $\forall x \forall y$, $x + y = y + x$

12. $\exists x \exists y$, $x^2 = y$ **13.** $\forall x \forall y$, $xy = yx$ **14.** $1 < 2 \rightarrow \exists x(x < 2)$ **15.** $\forall x\left[\sqrt{x} = k \rightarrow \exists y\left(\sqrt{y} = k\right)\right]$

16. $\forall f, D(f) \rightarrow C(f)$ **17.** $\exists f, C(f) \wedge \sim D(f)$

18. $[\forall x, E(x) \rightarrow A(x)] \wedge [\exists x, A(x) \wedge E(x)]$

19. $[\forall x, V(x) \rightarrow \sim O(x)] \rightarrow [\exists x, O(x) \wedge \sim V(x)]$

20. $[\forall x, E(x) \rightarrow L(x)] \rightarrow [\exists x, L(x) \wedge \sim E(x)]$ **21.** $\forall x, S(x) \rightarrow R(x)$ **22.** $\varnothing$; F

23. $\{\ldots, -3, -2, -1\}$, or $^{-}\mathbb{N}$; F **24.** $^{-}\mathbb{N}$; T **25.** $\{-1\}$; F **26.** R; T **27.** R; T **28.** $\varnothing$; F **29.** $^{-}\mathbb{N}$; T **30.** $^{-}\mathbb{N}$; T **31.** $\{-1\}$; T **32.** R; T **33.** R; T **34.** If $\forall x P(x)$ is true, then $\exists x P(x)$ is true. (Recall that we are considering only nonempty universal sets.) We will prove this later.

Exercise Set 1.7

1a. T **1b.** F; the sentence $\exists y(y < 0)$ is false. **1c.** F; each sentence $\forall x(0 < x)$, $\forall x(1 < x)$, $\forall x(2 < x)$ is false. **1d.** T; $y = 0$ **1e.** T; given an x there is a y, $y = x$, such that $y \le x$. **1f.** T; $\forall x(0 \le x)$ is true. **1g.** F; there is no number in the set which when added to 1 or 2 yields 0.

1h. T; this is the commutative law of addition. **1i.** F **2a.** T **2b.** F **2c.** F **2d.** T **2e.** T **2f.** T **2g.** F **2h.** T **2i.** T **3a.** T **3b.** T **3c.** F **3d.** F **3e.** T **3f.** F **3g.** T **3h.** T **3i.** T **4a.** F; the function f described by $f(x) = |x|$ is not differentiable at 0 and hence not differentiable. **4b.** T **4c.** T; there are many such functions; e.g., $f(x) = \sin x$. **4d.** T; see the answer to (a). **5a.** T **5b.** T **5c.** T; $\sum u_n = \sum n^{-1}$ **6a.** T **6b.** F **6c.** F **6d.** No, "$x < y$" yields a false sentence of this type. See (c). **6e.** T **6f.** Yes, we prove this later. **8a.** T **8b.** F **8c.** F **8d.** F **8e.** F **8f.** No **9.** Not every sentence of this type is true.

Exercise Set 1.8

1 – 10, 15, 16, 17 are tautologies. 11 – 14 are not.

Exercise Set 1.9

1. $3x = 15$ **2.** Q **3.** $P \vee Q$ **4.** Valid **5.** Valid **6.** Valid **7.** Valid **8.** Valid **9.** Valid

10. Valid **11.** Invalid **12.** Invalid **13.** Valid **14.** Invalid **15.** Valid **16.** Valid

Exercise Set 1.10

1. If $f'(x) \neq \cos x$, then $f(x) \neq \sin x$. Yes, because the previous sentence is true and its contrapositive is equivalent to it. **2.** x is not odd → x is even. **3.** x is not real → x is not rational. **4.** f discontinuous → f is not differentiable. **5.** $x \notin B \rightarrow x \notin A$ **6.** x even $\rightarrow x^2$ even

7. x odd $\rightarrow x^2$ odd **8.** x^2 odd $\rightarrow x$ odd **9.** $A = \varnothing \rightarrow A \cap B = \varnothing$ **10.** $x \neq 0 \rightarrow \sim \forall \varepsilon \left(|x| < \varepsilon \right)$

11. $x \neq y \rightarrow f(x) \neq f(y)$ **12.** $f(x) > f(y) \rightarrow x > y$ **13.** $f(x) \geq f(y) \rightarrow x \geq y$

14. $\sum_{n=1}^{\infty} u_n$ div. $\rightarrow \sum_{n=1}^{\infty} |u_n|$ div.

Exercise Set 1.11

1. $\forall x,\ x \geq 0 \vee \sim Q(x)$ **2.** $\exists x \forall y \exists z \exists q \forall j,\ x + y + z + q + j \neq 0$ **3.** $P \wedge \sim Q \wedge R$ **4.** $\forall x \forall y \exists z,$ $xz = y$ **5.** $\exists \varepsilon \forall \delta \exists x \left[|x - c| < \delta \wedge |f(x) - f(c)| \geq \varepsilon \right]$ **6.** $\exists \varepsilon \forall n \exists m \left(m > n \wedge |a_m - a| \geq \varepsilon \right)$

7. $x \in \mathbb{Q} \wedge y \in J \wedge x + y \notin J$ or $x \in \mathbb{Q} \wedge y \in J \wedge x + y \in \mathbb{Q}$ **8.** $P \wedge Q \wedge \sim R$ **9.** $x \notin A \vee x \notin B$

10. $x \notin A \wedge x \notin B$ **11.** $P_1 \wedge \ldots \wedge P_n \wedge \sim Q$ **12.** $(P \wedge \sim Q) \vee (Q \wedge \sim P)$

13. $\exists a > 0\ \exists b > 0\ \forall n \in \mathbb{N},\ na \leq b$ **14a.** $\forall x,\ f(-x) = f(x)$ **14b.** $\exists x,\ f(-x) \neq f(x)$

15a. $\forall x,\ f(-x) = -f(x)$ **15b.** $\exists x,\ f(-x) \neq -f(x)$ **16a.** $\forall x \forall y,\ f(x) = f(y)$

16b. $\exists x \exists y,\ f(x) \neq f(y)$ **17a.** $\exists p > 0 \forall x,\ f(x + p) = f(x)$ **17b.** $\forall p > 0 \exists x,\ f(x + p) \neq f(x)$

18a. $\forall x \forall y \left[x \leq y \rightarrow f(x) \geq f(y) \right]$ **18b.** $\exists x \exists y \left[x \leq y \wedge f(x) < f(y) \right]$

19a. $\forall x \forall y \left[x < y \rightarrow f(x) < f(y) \right]$ **19b.** $\exists x \exists y \left[x < y \wedge f(x) \geq f(y) \right]$

20a. $\forall x \forall y \left[x < y \rightarrow f(x) > f(y) \right]$ **20b.** $\exists x \exists y \left[x < y \wedge f(x) \leq f(y) \right]$

21a. $\forall x \forall y \left[f(x) = f(y) \rightarrow x = y \right]$ **21b.** $\exists x \exists y \left[f(x) = f(y) \wedge x \neq y \right]$

22a. $\forall y \in B\, \exists x \in A,\ f(x) = y$ **22b.** $\exists y \in B\, \forall x \in A,\ f(x) \neq y$

23a. $\forall x\, \forall \varepsilon > 0\, \exists \delta > 0,\ 0 < |x - x_0| < \delta \to |f(x) - L| < \varepsilon$

23b. $\exists x\, \exists \delta > 0\, \forall \delta > 0,\ 0 < |x - x_0| < \delta \wedge |f(x) - L| \geq \varepsilon$

24a. $\exists M\ \forall x, |f(x)| \leq M$ **24b.** $\forall M\ \exists x, |f(x)| > M$

25a. $\forall x\, \forall \varepsilon > 0\, \exists \delta > 0, |x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon$

25b. $\exists x\, \exists \varepsilon > 0\, \forall \delta > 0, |x - x_0| < \delta \wedge |f(x) - f(x_0)| \geq \varepsilon$

26a. $\forall x \in E\ \forall \varepsilon > 0\, \exists \delta > 0\ \forall y \in E, |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$

26b. $\exists x \in E\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists y \in E, |x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon$

27a. $\forall \varepsilon > 0\, \exists \delta > 0\ \forall x \in E\ \forall y \in E, |x - y| < \delta \to |f(x) - f(y)| < \varepsilon$

27b. $\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in E\ \exists y \in E, |x - y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon$

28a. $\forall \varepsilon > 0\, \exists n_0 \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall n \in \mathbb{N}, (m > n_0 \wedge n > n_0) \to |a_n - a_m| < \varepsilon$

28b. $\exists \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists m \in \mathbb{N}\ \exists n \in \mathbb{N}, (m > n_0 \wedge n > n_0) \to |a_n - a_m| \geq \varepsilon$

29. $\sum_{n=1}^{\infty} n^{-1}$ does not converge **30.** To show: $\exists \{u_n\}_{n=1}^{\infty},\ \lim_{n\to\infty} u_n = 0 \wedge \sum_{n=1}^{\infty} u_n$ does not converge. Consider $\{n^{-1}\}_{n=1}^{\infty};\ \lim_{n\to\infty} n^{-1} = 0 \wedge \sum_{n=1}^{\infty} n^{-1}$ does not converge. **31.** $f(x) = |x|$ provides a counterexample. **32.** $f(x) = x^2$ provides a counterexample.

Exercise Set 2.1

1. *a* = 2*k* + 1 iff *a* is odd. **2.** A polygon is a quadrilateral iff it has just four sides. **3.** $\max_S f$ is the maximum value of *f* on *S* iff it is the largest value assumed by *f* on *S*. **6.** *x* is a real number iff *x* equals an infinite decimal. **7.** *x* is irrational iff *x* is a real number which is not rational. **8.** A number is complex iff it is of the form *x* + *yi*, where *x* and *y* are real numbers and $i^2 = -1$.

Exercise Set 2.2

1. See text.

Exercise Set 2.3

1. a is an odd integer $\to a^2$ is an odd integer.

2. *Proof.* Assume a is an odd integer. Then $a = 2k + 1$ for some integer k, so $a^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Hence a^2 is an odd integer. ■

4. *Proof.* Assume a even and b even. Then there exist integers k and m such that $a = 2k$ and $b = 2m$. Then $ab = (2k)\cdot(2m) = 2(k2m)$, so ab is even. ■

6. Assume a even and b odd. Then there exist integers k and m such that $a = 2k$ and $b = 2m + 1$. Then $ab = 2k(2m+1)$, so ab is even. ■

10. *Proof.* Assume $\exists y \forall x, P(x,y)$. There is a replacement b for y such that $\forall x, P(x,b)$ is true. Thus for every u in the universal set $P(u,b)$ is true. Then b yields a true sentence for every u, so

$\exists y, P(u,y)$ is true for every u. Hence, $\forall x \exists y, P(x,y)$ is true. ■

13. $[P \to (Q \wedge R)] \leftrightarrow [(P \to Q) \wedge (P \to R)]$ **15.** $[(P \to Q) \to (S \to R)] \leftrightarrow$ $[((P \to Q) \wedge S) \to R]$

Exercise Set 2.4

1. Proof has been given before. **3a.** (→) (This holds by O3 of the Appendix.

3b. (←) *Proof.* Assume $a + c < b + c$. Then by A4 and O3 in the Appendix, $(a+c)+(-c) < (b+c)+(-c)$; by A2, A3, and A4,

$$a + [c + (-c)] < b + [c + (-c)]$$
$$a + 0 < b + 0$$
$$a < b.$$

■

4a. *Proof.* Assume x is odd. Then for some integer k, $x = 2 \cdot k + 1$. Then $x + 1 = 2k + 2 =$

$2(k + 1)$, so $x + 1$ is even. ■

4b. *Proof*. Assume x + 1 is even. Then for some k, $x+1=2\cdot k$, thus

$$\begin{aligned} x &= 2\cdot k-1 \\ &= 2k-1+2-2 \\ &= 2k-2+1 \\ &= 2(k-1)+1; \end{aligned}$$

so x is odd. ■

8. $\left[P\to(Q\leftrightarrow R)\right]\leftrightarrow\left[(P\wedge Q)\to R\wedge(P\wedge R)\to Q\right]$

Exercise Set 2.5

1. Let x be arbitrary. Prove "x^2 is even iff x is even" by proving a) if x^2 is even, then x is even, and b) if x is even, then x^2 is even. **8.** To do this let x be arbitrary and prove $x + y = x$.

Exercise Set 2.6

1. *A* right; *A* obtuse **2a.** *f* is not differentiable. **2b.** *f* is odd. **2c.** *f* is not constant. **3a.** x is odd. **3b.** $x = 9$

4. *Proof*. x real $\to x>0\vee x<0\vee x=0$.

CASE 1)

$$\begin{aligned} &x>0\to|x|=x \\ &x>0\to -x<0\to|-x|=-(-x)=x \\ &\therefore|-x|=x \end{aligned}$$

CASE 2)

$$\begin{aligned} &x<0\to|x|=-x \\ &x<0\to -x>0\to|-x|=-x \\ &\therefore|-x|=|x| \end{aligned}$$

CASE 3)

$$\begin{aligned} &x=0\to|x|=x \\ &x=0\to -x=0\to|-x|=0=x \\ &\therefore|-x|=|x| \end{aligned}$$

■

5. *Proof.* x real $\to x \geq 0 \vee x < 0$.

CASE 1)

$$x \geq 0 \to x^2 \geq 0 \to \left|x^2\right| = x^2$$
$$x \geq 0 \to |x| = x \to |x|^2 = x^2$$
$$\therefore |x|^2 = \left|x^2\right|$$

CASE 2)

$$x < 0 \to x^2 > 0 \to \left|x^2\right| = x^2$$
$$x < 0 \to |x| = -x \to |x|^2 = (-x)(-x) = x^2$$
$$\therefore \left|x^2\right| = |x|^2$$

■

13. *Proof.* x is an integer $\to$ x is even or x is odd.

CASE 1) x is even. Then $x = 2k$ for some $k \in \mathbb{Z}$. Hence

$$\begin{aligned} x^2 - x &= (2k)^2 - (2k) \\ &= 4k^2 - 2k \\ &= 2\left(2k^2 - k\right), \end{aligned}$$

so $x^2 - x$ is even.

CASE 2) x is odd. Then $x = 2k + 1$ for some $k \in \mathbb{Z}$. You complete. ■

14. Analogous to Exercise 13.

18. *Proof.* $x \neq 0 \to x > 0$ or $x < 0$.

CASE 1) $x > 0 \to f(x) = |x| = x \to f'(x) = 1.$

CASE 2) $x < 0 \to f(x) = |x| = -x \to f'(x) = -1.$ ■

Exercise Set 2.7

In most of these answers only the induction step is given. You should be able to complete the basis step.

1. Assume $P(k): 2^k > k$

Deduce $P(k+1): 2^{k+1} > k+1$

Now $2^{k+1} = 2 \cdot 2^k > 2k$, by $P(k)$

$2k = k + k \geq k + 1$, since for every natural number k, $k \geq 1$

We used Property O3 of the Appendix.

2. ANALYSIS: Analogous to Exercise 1.

3. Assume $P(k): 2 \leq 2^k$

Deduce $P(k+1): 2 \leq 2^{k+1}$

$$\begin{aligned} \text{Now } 2^{k+1} &\geq 2^k, \text{ by an example} \\ &\geq 2, \text{ by } P(k) \end{aligned}$$

4. Assume $P(k): 2k \leq 2^k$

Deduce $P(k+1): 2(k+1) \leq 2^{k+1}$

$$\begin{aligned} \text{Now } 2(k+1) = 2k + 2 &\leq 2^k + 2, \text{ by } P(k) \\ &\leq 2^k + 2^k, \text{ by Exercise 3} \\ &= 2(2^k) = 2^{k+1} \end{aligned}$$

6. Assume $P(k): 2^{k-1} \leq k!$

Deduce $P(k+1): 2^k \leq (k+1)!$

$$\begin{aligned} \text{Now } 2^k = 2(2^{k-1}) &\leq 2 \cdot k!, \text{ by } P(k) \\ &\leq (k+1)k!, \text{ since } k \geq 1 \rightarrow k+1 \geq 1+1 = 2 \\ &= (k+1)! \end{aligned}$$

7. BASIS STEP.

$P(4): 2^4 < 4!$

Now $2^4 = 16$ and $4! = 24$ so $2^4 < 4!$

INDUCTION STEP.

Assume $P(k): 2^k < k!$

Deduce $P(k+1): 2^{k+1} < (k+1)!$

$$\begin{aligned} 2^{k+1} = 2\left(2^k\right) &< 2 \cdot k!, \text{ by } P(k) \\ &< (k+1)k! \\ &= (k+1)! \end{aligned}$$

9. INDUCTION STEP.

Assume $P(k): (2k)! < 2^{2k}(k!)^2$

Deduce $P(k+1): \left[2(k+1)\right]! < 2^{2k+2}\left[(k+1)!\right]^2$

Now $\left[2(k+1)\right]! = (2k+2)(2k+1)(2k)! = \left(4k^2+6k+2\right)(\boldsymbol{2k})!$

$$\begin{aligned} \text{and } 2^{2k+2}\left[(k+1)!\right]^2 &= 2^{2k+2} \cdot \left[(k+1)(k!)\right]^2 \\ &= 2^{2k+2} \cdot (k+1)^2 (k!)^2 \\ &= 2^{2k} \cdot 2^2 \cdot \left(k^2+2k+1\right)(k!)^2 \\ &= \mathbf{2^{2k}}\left(4k^2+8k+4\right)(k!)^2 \end{aligned}$$

Then noting that $\left(4k^2+6k+2\right) < 4k^2+8k+4$ and using $P(k)$ we have

$$\begin{aligned} \left[2(k+1)\right]! &= \left(4k^2+6k+2\right)(2k)! \\ &< \left(4k^2+6k+2\right) \cdot 2^{2k} \cdot (k!)^2, \text{ by } P(k) \\ &< 2^{2k} \cdot \left(4k^2+8k+4\right)(k!)^2 \\ &= 2^{2k+2}\left[(k+1)!\right]^2 \end{aligned}$$

ANALYSIS: The two expressions of $P(k + 1)$ were expanded hopefully to see where $P(k)$ could be applied. Note the bold type. The proof then followed via $P(k)$ and the noted inequality.

10. INDUCTION STEP.

Assume $P(k): |\sin kx| \le k|\sin x|$

Deduce $P(k+1): |\sin(k+1)x| \le (k+1)|\sin x|$

Now

$$\begin{aligned}|\sin(k+1)x| &= |\sin(kx+x)| \\ &= |\sin kx\cos x+\cos kx\sin x|, \text{ by the formula in the hint} \\ &\le |\sin kx\cos x|+|\cos kx\sin x|, \text{ by } |p+q|\le|p|+|q| \\ &= |\sin kx|\cdot|\cos x|+|\cos kx|\cdot|\sin x|, \text{ since } |pq|=|p|\cdot|q| \\ &\le |\sin kx|+|\sin x|, \text{ since } |\cos x|\le 1 \text{ and } |\sin x|\le 1 \\ &\le k|\sin x|+|\sin x|, \text{ by } P(k) \\ &= (k+1)|\sin x|.\end{aligned}$$

13. BASIS STEP.

Prove $P(2):\cos u=\dfrac{\sin 2u}{2\sin u}$

$$\frac{\sin 2u}{2\sin u}=\frac{2\sin u\cos u}{2\sin u}=\cos u$$

INDUCTION STEP.

Assume $P(k):\cos u\cdots\cos 2^{k-1}u=\dfrac{\sin 2^k u}{2^k\sin u}$

Deduce $P(k+1):\cos u\cdots\cos 2^{k-1}u\cos 2^k u=\dfrac{\sin 2^{k+1}u}{2^{k+1}\sin u}$

Now

$$\begin{aligned}\cos u\cdots\cos 2^{k-1}u\cdot\cos 2^k u &= \frac{\sin 2^k u}{2^k\sin u}\cdot\cos 2^k u \text{ by } P(k) \\ &= \frac{\sin 2^k u\cdot\cos 2^k u}{2^k\sin u} \\ &= \frac{\frac{1}{2}\sin 2\left(2^k u\right)}{2^k\sin u}, \text{ since } 2\sin a\cos a=\sin 2a \\ &= \frac{\sin 2^{k+1}u}{2^{k+1}\sin u}\end{aligned}$$

19. INDUCTION STEP.

Assume $P(k): \sum_{j=1}^{k} j = \frac{k^2+k}{2}$

Deduce $P(k+1): \sum_{j=1}^{k+1} j = \frac{(k+1)^2+(k+1)}{2} = \frac{k^2+3k+2}{2}$

Now

$$\begin{aligned}\sum_{j=1}^{k+1} j &= \sum_{j=1}^{k} j + (k+1), \text{ by definition of } \Sigma\text{-notation}\\ &= \frac{k^2+k}{2} + (k+1), \text{ by } P(k)\\ &= \frac{k^2+k}{2} + \frac{2(k+1)}{2}\\ &= \frac{k^2+3k+2}{2}\end{aligned}$$

32. Test the argument when proving $P(1) \to P(2)$. **49e.** $2^n - 1$

Exercise Set 2.8

1. Analogous to EXAMPLE 2. **2.** Assume for contradiction that there exist an x and y such that $x \neq 0 \wedge y \neq 0 \wedge xy = 0$. Then $x^{-1} \cdot (xy) = (x^{-1} \cdot x) y = 1 \cdot y = y$, by M4, M5. Also, since $xy = 0$, $x^{-1} \cdot (xy) = x^{-1} \cdot 0 = 0$. Therefore $y = 0$. But $y \neq 0$ is assumed. **3.** Assume for contradiction that there exists an x such that $x > 0 \wedge x^{-1} \leq 0$. Then $x \cdot x^{-1} \leq x \cdot 0$, by O3. But $x \cdot x^{-1} = 1$ and $x \cdot 0 = 0$. Therefore $1 \leq 0$. But by O7, $1 > 0$. **5.** Assume for contradiction that there exists an x such that $x > 0, \sqrt{x} \geq \sqrt{x+1}$. Then

$$\begin{aligned}x = \sqrt{x} \cdot \sqrt{x} &\geq \sqrt{x} \cdot \sqrt{x+1}, \text{ by O3 since } \sqrt{x} > 0\\ &\geq \sqrt{x+1} \cdot \sqrt{x+1}, \text{ by assumption}\\ &= x+1\end{aligned}$$

Therefore $x \geq x+1$, which contradicts the hint. **7a.** *Hint:* Having negated the sentence use the fact that $\sqrt{2}$ is irrational and consider $a = \left(\sqrt{2}\right)^{\sqrt{2}}$ and $b = \sqrt{2}$. **7b.** No **8.** Since for every real number x, $x + k = k + x = x$, then $0 + k = k + 0 = 0$. But we also know that $k + 0 = k$. Therefore $k = 0$. Hence $k \neq 0$ and $k = 0$, which is a contradiction. **14.** Assume for contradiction that there exists an $x > 0$ such that for every even number m, $m \leq x$. This means that every even number is less than or equal to x. But consider $2x$. This is an even number such that $x < 2x$, since $x > 0$. This contradicts every even number being less than x.

Exercise Set 2.9

1. $\exists! \ell, P \in \ell \wedge Q \in \ell$ **2.** Same as 1 **3.** $\exists! x \forall y, x + y = y + x = y$ **4.** $\exists! x \forall y, x \cdot y = y \cdot x = y$

5. $\forall x \forall y \exists! z, x + y = z$ **6.** $\forall x \exists! y, x + y = y + x = 0$ **8.** Analogous to the example.

9. Existence: See A4 of the Appendix. Uniqueness: Let x be arbitrary. Assume there are two elements z and y such that $x + y = y + x = 0$ and $x + z = z + x = 0$. Then

$$\begin{aligned}(y+x)+z &= z, \text{ since } 0+z=z \\ y+(x+z) &= z, \text{ associative law A2} \\ y+0 &= z, \text{ since } x+z=0 \\ y &= z, \text{ since } y+0=y\end{aligned}$$

Exercise Set 2.10

1. Assume: $\sim Q$

Deduce: $\sim P$

Analytic Process:

$$\begin{aligned}&\sim P \text{ if } R\left(R \to \sim P\right) \\ &R \text{ if } S\left(S \to R\right) \\ &S \text{ if } \sim Q\left(\sim Q \to S\right) \\ &\text{Hence } \sim Q \to S \to R \to \sim P \\ &\therefore \sim Q \to \sim P\end{aligned}$$

3. e **4.** f **5.** a **6.** d **7.** c **8.** b **9.** g **10.** h **11.** $x = 5$ **12.** $\delta = \varepsilon/3$

Exercise Set 3.1

Brief proofs or hints are provided, often just an analysis.

1. ANALYSIS: See Theorem 4a). **3.** ANALYSIS: See Theorem 8.

4.

$$\left(A' \cup B'\right)' = \left(A'\right)' \cap \left(B'\right)', \text{ by Thm. 8}$$
$$= A \cap B, \text{ by Thm. 5}$$
$$\text{Then by Thm. 7,}$$
$$\left[\left(A' \cup B'\right)'\right]' = \left(A \cap B\right)', \text{ and by Thm. 5,}$$
$$A' \cup B' = \left(A \cap B\right)'.$$

7a). $x \in A \to x \in A,$ by the tautology $P \to P$

9.

$$A \subseteq B \leftrightarrow A \cap B' = \varnothing, \text{ by Theorem 6}$$
$$\leftrightarrow \left(A \cap B'\right)' = \varnothing', \text{ by Theorem 7}$$
$$\leftrightarrow A' \cup \left(B'\right)' = U, \text{ by Theorem 9, and Exercise 8}$$
$$\leftrightarrow A' \cup B = U, \text{ by Theorem 5}$$

10. ANALYSIS. Note the similarity of $A \cup \left(B \cup C\right) = \left(A \cup B\right) \cup C$ to $P \vee \left(Q \vee R\right) \leftrightarrow \left(P \vee Q\right) \vee R$

12. $A \subseteq A \cup \varnothing$ by Theorem 11. To prove $A \cup \varnothing \subseteq A$ note that

$$x \in A \cup \varnothing \to x \in A \vee x \in \varnothing, \text{ by Definition 2}$$
$$\to x \in A, \text{ since } x \in \varnothing \text{ is false}$$

13. $x \in A \vee x \notin A,$ true by the tautology $P \vee \sim P$

$\to x \in A \vee x \in A',$ for every x

$\to x \in A \cup A',$ for every x

$\to \forall x, x \in A \cup A'$

$\to A \cup A' = U,$ by Axiom 2

15. *Hint:* Use Exercise 8 and Theorems 5 and 7.

17. To prove $\forall x(x \in A \rightarrow x \in U)$. Consider

$$\underbrace{\underset{\text{F}}{x \in A} \rightarrow \underset{\text{always true}}{x \in U}}_{\text{T, by definition of '}\rightarrow\text{'}}$$

$$\underbrace{\underset{\text{T}}{x \in A} \rightarrow \underset{\text{always true}}{x \in U}}_{\text{T, by definition of '}\rightarrow\text{'}}$$

20. ANALYSIS. Use Theorem 12 and Theorem 5.

23.

$$\begin{aligned} A \cup B = B &\leftrightarrow \forall x(x \in A \cup B \leftrightarrow x \in B), \text{ by Axiom 1} \\ &\leftrightarrow \forall x\left[(x \in A \vee x \in B) \leftrightarrow x \in B\right], \text{ by Definition 2} \\ &\leftrightarrow \forall x(x \in A \rightarrow x \in B), \text{ by the tautology } \left[(P \vee Q) \leftrightarrow Q\right] \leftrightarrow (P \rightarrow Q) \\ &\leftrightarrow A \subseteq B, \text{ by Definition 1.} \end{aligned}$$

24. ANALYSIS: Analogous to Exercise 23. **25.** ANALYSIS: Use Theorems 2 and 3.

28. $A - \varnothing = A \cap \varnothing' = A \cap U = A$

30.

$$\begin{aligned} B-(B-A) &= B \cap (B \cap A')' \\ &= B \cap (B' \cup A) \\ &= (B \cap B') \cup (B \cap A) \\ &= \varnothing \cup (B \cap A) \\ &= B \cap A \end{aligned}$$

Now $A \subseteq B \rightarrow B \cap A = A.$ Hence $A \subseteq B \rightarrow B-(B-A) = A.$

34. Let $A = \{1,2,3\}$ and $B = \{2,3,4\}$.

Exercise Set 3.2

2a. A_1 **2b.** A_n **3.** See Theorem 14 for an analogous proof. **4a.** A_1 **4b.** A_n **4c.** A_1 **4d.** $\varnothing$

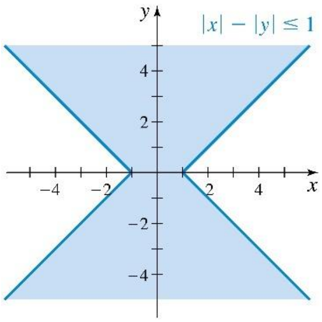
6a. A_6 **6d.** $\mathbb{N}$ **6e.** $\varnothing$ **10a.** $\varnothing$ **10b.** Yes **10c.** $\{3,4\}, \varnothing, \varnothing$ **10d.** No **10e.** The sets in this exercise provide a counterexample **10f.** True **11.** Prove the contrapositive, $A = \varnothing$ and $B = \varnothing \rightarrow A \cup B = \varnothing$. Assume $A = \varnothing$ and $B = \varnothing$. Then $A \cup B = \varnothing \cup \varnothing = \varnothing$, by Exercise 12, Exercise Set 3.1 **12.** Suppose, for contradiction, that there exists a subset A of U, for which

$A = A'$. Recall that the universal set is non-empty. Thus, there exists at least one element $x \in U$. Now $U = A \cup A'$, so if x is in U it must be in A or in A'. CASE 1. Suppose $x \in A$. By definition, $x \notin A'$, so $A \neq A'$. CASE 2. Suppose $x \in A'$. By definition $x \notin A$, so $A \neq A'$. We have a contradiction in each case.

Exercise Set 3.3

1. $\{(a,5),(a,6),(b,5),(b,6),(c,5),(c,6)\}$ **7.** $(4,2),(4,-2),\left(3,\sqrt{3}\right)$ **8.** $D_M = \{x \mid x \geq 0\}$, $R_M = \mathbb{R}$ **9a.** An ellipse with vertices at $(0,5),(0,-5),(2,0),(-2,0)$ **9b.** $\{x \mid -2 \leq x \leq 2\}$, $\{y \mid -5 \leq y \leq 5\}$ **11.** Assume $(a,b) = (b,a)$. Then $\{\{a\},\{a,b\}\} = \{\{b\},\{b,a\}\}$. Then $\{a\} = \{b\}$, so $a = b$. **12.** All false except b **20.** Line containing points $(0,-1)$ and $(1,2)$

21. $|x| - |y| \leq 1$



23. Exterior of circle centered at $(0,0)$ with radius 2. **27.** $\rho^{-1} = \{(x,y) \mid x = 3y - 1\}$

32. $\rho = \rho^{-1}$

Exercise Set 3.4

2. ρ is symmetric $\leftrightarrow \forall a, \forall b \in A, (a,b) \in \rho \rightarrow (b,a) \in \rho$; ρ is <u>not</u> symmetric $\leftrightarrow \exists a\ \exists b \in A, (a,b) \in \rho \wedge (b,a) \notin \rho$ **4.** 6, 9, 10, 11, 12, 14 **5.** 6, 7, 8, 10, 11, 12, 14 **6.** 6, 9, 10, 11, 12, 13, 14 **7.** 6, 10, 11, 12, 14 **9.** $E(T) =$ the set of all triangles congruent to T, for each triangle T

Exercise Set 3.5

1. For any integer *a*, $a-a=0=3\cdot 0$, so $(a,a)\in\rho$, and ρ is reflexive. Assume for any fixed but arbitrary integers *a*, *b*, that $(a,b)\in\rho$. Then $a-b=3k$ for some integer *k*. Then $b-a=3(-k)$, and since $-k$ is still an integer, $(b,a)\in\rho$, so ρ is symmetric. Assume $(a,b)\in\rho$ and $(b,c)\in\rho$. Then $a-b=3k$ and $b-c=3m$, for some integers *k* and *m*. Then $a-c=(a-b)+(b-c)=3k+3m=3(k+m)$, so ρ is transitive. **7.** You don't know at the outset that $(a,b)\in\rho$ for any elements *a*, *b*.

8. c); $\{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(a,c),(c,a),(b,c),(c,b),(d,e),(e,d)\}$

Exercise Set 3.6

1 and 3 are functions; the rest are not. 7 – 11 are all functions. **7.** $D_f=\{a,b,c\}$,

$R_f=\{1,2,3\}$ **11.** $D_f=\{a,b,c,d\}$, $R_f=\{1,3,5\}$ **12a.** $f:A\to B$ is **onto** $\leftrightarrow \forall b\in B\ \exists a\in A,$ $f(b)=a.$ **12b.** *f* is <u>not</u> onto $\leftrightarrow \exists b\in B\ \forall a\in A,\ f(a)\neq b.$ **13.** None

14. None **15a.** $f:A\to B$ is **one-to-one** $\leftrightarrow \forall a\in A\ \forall b\in B,\ f(a)=f(b)\to a=b.$

15b. *f* is <u>not</u> one-to-one $\leftrightarrow \exists a\in A\ \exists b\in B,\ f(a)=f(b)\wedge a\neq b.$ **16.** Neither **17.** 7, 8, 9

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